

WEAK ORDERS ON SYMMETRIC GROUPS AND POSETS OF SUPPORT τ -TILTING MODULES

RYOICHI KASE

ABSTRACT. We give a necessary and sufficient condition for that the support τ -tilting poset of a finite dimensional algebra Λ is isomorphic to the poset of symmetric group \mathfrak{S}_{n+1} with weak order. Moreover we show that there are infinitely many finite dimensional algebras whose support τ -tilting posets are isomorphic to \mathfrak{S}_{n+1} .

1. INTRODUCTION

The notion of tilting modules was introduced in [BrB]. It is known that they control derived equivalence [H]. Therefore to obtain many tilting modules is an important problem in representation theory of finite dimensional algebras. Tilting mutation given by Riedtmann-Schofield [RS] is an approach to this problem. It is an operation which gives a new tilting module from given one by replacing an indecomposable direct summand. However tilting mutation is not always possible depending on a choice of an indecomposable direct summand.

Adachi-Iyama-Reiten introduced the notion of support τ -tilting modules as a generalization of tilting modules [AIR]. They give a mutation of support τ -tilting modules and complemented that of tilting modules. i.e. the support τ -tilting mutation has following nice properties:

- Support τ -tilting mutation is always possible.
- There is a partial order on the set of (isomorphism classes of) basic support τ -tilting modules such that its Hasse quiver realizes the support τ -tilting mutation. (An analogue of Happel-Unger's result [HU] for tilting modules.)

Moreover they showed deep connections between τ -tilting theory, silting theory, torsion theory and cluster tilting theory.

Then for several classes of algebras, support τ -tilting posets are calculated. One interesting example is a preprojective algebra of Dynkin type. Preprojective algebras play an important role in representation theory of algebras and Lie theory. Mizuno shows the following result.

Theorem 1.1. [M, Theorem 2.30] *Let Λ be a preprojective algebra of Dynkin type. Then the support τ -tilting poset of Λ is isomorphic to corresponding Weyl group with weak order.*

In particular, the support τ -tilting poset of preprojective algebra of type A is realized by the symmetric group with weak order. Such an algebra is not only preprojective algebra of type A . Iyama-Zhang shows that support τ -tilting poset of the Auslander algebra of the truncated polynomial ring is also isomorphic to the symmetric group with weak order [IZ]. In this paper we classify such algebras.

Notation. Throughout this paper, let $\Lambda = kQ/I$ be a basic finite dimensional algebra over an algebraically closed field k , where Q is a finite quiver and I is an admissible ideal of kQ .

We denote by Q_0 the set of vertices of Q and Q_1 the set of arrows of Q . We set Q° the quiver obtained from Q by deleting all loops.

1. For arrows $\alpha : a_0 \rightarrow a_1$ and $\beta : b_0 \rightarrow b_1$ of Q , we mean by $\alpha\beta$ the path $a_0 \xrightarrow{\alpha} a_1 \xrightarrow{\beta} b_1$ if $a_1 = b_0$, otherwise 0 in kQ .
2. We denote by $\text{mod } \Lambda$ ($\text{proj } \Lambda$) the category of finitely generated (projective) right Λ -modules.
3. By a module, we always mean a finitely generated right module.
4. For a poset \mathbb{P} and $a, b \in \mathbb{P}$, we denote by $\mathcal{H}(\mathbb{P})$ the Hasse quiver of \mathbb{P} and put $[a, b] := \{x \in \mathbb{P} \mid a \leq x \leq b\}$. We denote by $\text{dp}(a)$ the set of direct predecessor of a in $\mathcal{H}(\mathbb{P})$ and by $\text{ds}(a)$ the set of direct successor of a in $\mathcal{H}(\mathbb{P})$. We say that \mathbb{P} is *n-regular* provided $\text{dp}(a) + \text{ds}(a) = n$ holds for any element $a \in \mathbb{P}$. We call a subposet \mathbb{P}' of \mathbb{P} a *full subposet* if the inclusion $\mathbb{P}' \subset \mathbb{P}$ induces a quiver inclusion from $\mathcal{H}(\mathbb{P}')$ to $\mathcal{H}(\mathbb{P})$. By definition if \mathbb{P}' is a full subposet of \mathbb{P} , then $\mathcal{H}(\mathbb{P}')$ is a full subquiver of $\mathcal{H}(\mathbb{P})$.

2. PRELIMINARY

In this section, we recall the definitions and their basic properties of support τ -tilting modules, silting complexes and the weak order on Symmetric groups.

2.1. Support τ -tilting modules. For a module M , we denote by $|M|$ the number of non-isomorphic indecomposable direct summands of M . The Auslander-Reiten translation is denoted by τ . (Refer to [ASS, ARS] for definition and properties.)

Let us recall the definition of support τ -tilting modules.

Definition 2.1. Let M be a Λ -module and P a projective Λ -module.

- (1) We say that M is *τ -rigid* if it satisfies $\text{Hom}_\Lambda(M, \tau M) = 0$.
- (2) A pair (M, P) is said to be *τ -rigid* if M is τ -rigid and $\text{Hom}_\Lambda(P, M) = 0$.
- (3) A support τ -tilting pair (M, P) is defined to be a τ -rigid pair with $|M| + |P| = |\Lambda|$.
- (4) We call M a *support τ -tilting module* if there exists a projective module P such that (M, P) is a support τ -tilting pair. The set of isomorphism classes of basic support τ -tilting modules of Λ is denoted by $\text{s}\tau\text{-tilt } \Lambda$.

We denote by e_i the primitive idempotent corresponding to a vertex i of Q . For a module M , we define a subset of Q_0 by

$$\text{Supp}(M) := \{i \in Q_0 \mid Me_i \neq 0\}.$$

If $(M, e\Lambda)$ is a support τ -tilting pair for some idempotent e , then $\text{Supp}(M)$ coincides with the set of vertices i satisfying $ee_i = 0$.

Proposition 2.2. [AIR, Proposition 2.3] *Let M be a support τ -tilting module. If (M, P) and (M, P') are support τ -tilting pairs, then $\text{add } P = \text{add } P' = \text{add } e\Lambda$, where $e = \sum_{i \in Q_0 \setminus \text{Supp}(M)} e_i$.*

Proposition 2.3. [AIR, Proposition 1.3, Lemma 2.1] *The following hold.*

- (1) *A τ -rigid pair (M, P) satisfies the inequality $|M| + |P| \leq |\Lambda|$.*

- (2) Let J be an ideal of Λ . Let M and N be (Λ/J) -modules. If $\text{Hom}_\Lambda(M, \tau N) = 0$, then $\text{Hom}_{\Lambda/I}(M, \tau_{\Lambda/J} N) = 0$. Moreover, if $J = (e)$ is a two-sided ideal generated by an idempotent e , then the converse holds.

Denote by $\text{Fac } M$ the category of factor modules of finite direct sums of copies of M .

Definition-Theorem 2.4. [AIR, Lemma 2.25] For support τ -tilting modules M and M' , we write $M \geq M'$ if $\text{Fac } M \supseteq \text{Fac } M'$. Then one has the following equivalent conditions:

- (1) $M \geq M'$.
- (2) $\text{Hom}_\Lambda(M', \tau M) = 0$ and $\text{Supp}(M) \supseteq \text{Supp}(M')$.

Moreover, \geq gives a partial order on $\text{s}\tau\text{-tilt } \Lambda$.

Let (N, R) be a pair of a module N and a projective module R .

We say that (N, R) is *basic* if so are N and R . A direct summand (N', R') of (N, R) is also a pair of a module N' and a projective module R' which are direct summands of N and R , respectively.

A pair (N, R) is said to be *almost complete support τ -tilting* provided it is a τ -rigid pair with $|N| + |R| = |\Lambda| - 1$.

Theorem 2.5. (1) [AIR, Theorem 2.18] *Every basic almost complete support τ -tilting pair is a direct summand of exactly two basic support τ -tilting pairs.*

- (2) [AIR, Corollary 2.34] *Let (M, P) and (M', P') be basic support τ -tilting pairs. Then M and M' are connected by an arrow of $\mathcal{H}(\text{s}\tau\text{-tilt } \Lambda)$ if and only if (M, P) and (M', P') have a common basic almost complete support τ -tilting pair as a direct summand. In particular, $\text{s}\tau\text{-tilt } \Lambda$ is $|\Lambda|$ -regular.*

- (3) [AIR, Corollary 2.38] *If $\mathcal{H}(\text{s}\tau\text{-tilt } \Lambda)$ has a finite connected component \mathcal{C} , then $\mathcal{C} = \mathcal{H}(\text{s}\tau\text{-tilt } \Lambda)$.*

For a basic τ -rigid pair (N, R) , we define

$$\text{s}\tau\text{-tilt}_{N \oplus R^-} \Lambda := \{M \in \text{s}\tau\text{-tilt } \Lambda \mid N \in \text{add } M, \text{Hom}_\Lambda(R, M) = 0\},$$

equivalently, which consists of all support τ -tilting pairs having (N, R) as a direct summand. For simplicity, we omit 0 if $N = 0$ or $R = 0$.

Given an idempotent $e = e_{i_1} + \cdots + e_{i_\ell}$ of Λ so that $R = e\Lambda$, we see that M belongs to $\text{s}\tau\text{-tilt}_{N \oplus R^-} \Lambda$ if and only if it is a basic support τ -tilting module with $\text{Supp}(M) = Q_0 \setminus \{i_1, \dots, i_\ell\}$. Hence, by Proposition 2.3 this leads to a poset isomorphism $\text{s}\tau\text{-tilt}_{N \oplus R^-} \Lambda \simeq \text{s}\tau\text{-tilt } \Lambda / (e)$. More generally, we have following reduction theorem.

Theorem 2.6. [J] *Let (N, R) be a basic τ -rigid pair and let T be the Bongartz completion of (N, R) . If we set $\Gamma := \text{End}_\Lambda(T)/(e)$, then $|\Gamma| = |\Lambda| - |N| - |R|$ and $\text{s}\tau\text{-tilt}_{N \oplus R^-}(\Lambda) \simeq \text{s}\tau\text{-tilt}(\Gamma)$, where e is the idempotent corresponding to the projective $\text{End}_\Lambda(T)$ -module $\text{Hom}_\Lambda(T, N)$.*

Theorem 2.6 implies that for an idempotent $e \in \Lambda$, we have a poset isomorphism $\text{s}\tau\text{-tilt}_{e\Lambda} \Lambda \simeq \text{s}\tau\text{-tilt } \Lambda / (e)$.

2.2. Silting complexes. We denote by $\text{K}^b(\text{proj } \Lambda)$ the bounded homotopy category of $\text{proj } \Lambda$.

A complex $T = [\cdots \rightarrow T^i \rightarrow T^{i+1} \rightarrow \cdots]$ in $\text{K}^b(\text{proj } \Lambda)$ is said to be *two-term* provided $T^i = 0$ unless $i = 0, -1$.

We recall the definition of silting complexes.

Definition 2.7. Let T be a complex in $\mathbf{K}^b(\text{proj } \Lambda)$.

- (1) We say that T is *presilting* if $\text{Hom}_{\mathbf{K}^b(\text{proj } \Lambda)}(T, T[i]) = 0$ for any positive integer i .
- (2) A *silting complex* is defined to be presilting and generate $\mathbf{K}^b(\text{proj } \Lambda)$ by taking direct summands, mapping cones and shifts.

We denote by $\text{silt } \Lambda$ ($2\text{silt } \Lambda$) the set of isomorphism classes of basic (two-term) silting complexes in $\mathbf{K}^b(\text{proj } \Lambda)$.

We give an easy property of (pre)silting complexes.

Lemma 2.8. [AI, Lemma 2.25] *Let M be a τ -rigid module and $P_1 \xrightarrow{d} P_0 \rightarrow M \rightarrow 0$ a minimal projective presentation of M . Then $\text{add } P_1 \cap \text{add } P_0 = \{0\}$.*

Remark 2.9. Let $[P_1 \xrightarrow{d} P_0] \in 2\text{silt } \Lambda$. By Theorem 2.11 and Lemma 2.8, we may assume that $\text{add } P_1 \cap \text{add } P_0 = \{0\}$.

The set $\text{silt } \Lambda$ also has poset structure as follows.

Definition-Theorem 2.10. [AI, Theorem 2.11] For silting complexes T and T' of $\mathbf{K}^b(\text{proj } \Lambda)$, we write $T \geq T'$ if $\text{Hom}_{\mathbf{K}^b(\text{proj } \Lambda)}(T, T'[i]) = 0$ for every positive integer i . Then the relation \geq gives a partial order on $\text{silt } \Lambda$.

The following result connects silting theory with τ -tilting theory.

Theorem 2.11. [AIR, Corollary 3.9] *We consider an assignment*

$$\mathbf{S} : (M, P) \mapsto \begin{bmatrix} (-1\text{th}) & (0\text{th}) \\ P_1 \oplus P & \xrightarrow{(p_M, 0)} P_0 \end{bmatrix}$$

where $p_M : P_1 \rightarrow P_0$ is a minimal projective presentation of M .

- (1) [AIR, Lemma 3.4] *For modules M, N , the following are equivalent:*

- (a) $\text{Hom}_\Lambda(M, \tau N) = 0$.
- (b) $\text{Hom}_{\mathbf{K}^b(\text{proj } \Lambda)}(\mathbf{S}(M), \mathbf{S}(N)[1]) = 0$.

- (2) [AIR, Lemma 3.5] *For any projective module P and any module M , the following are equivalent:*

- (a) $\text{Hom}_\Lambda(P, M) = 0$.
- (b) $\text{Hom}_{\mathbf{K}^b(\text{proj } \Lambda)}(\mathbf{S}(0, P), \mathbf{S}(M)[1]) = 0$.

In particular, the assignment \mathbf{S} gives rise to a poset isomorphism $\text{s}\tau\text{-tilt } \Lambda \xrightarrow{\sim} 2\text{silt } \Lambda$.

In the end of this subsection, we recall g-vector of 2-term objects of $\mathbf{K}^b(\text{proj } \Lambda)$.

Definition 2.12. Let $X = [P' \rightarrow P]$ be a 2-term object of $\mathbf{K}^b(\text{proj } \Lambda)$. If $[P] - [P'] = \sum_{i \in Q_0} g_i [e_i \Lambda]$ in the Grothendieck group $K_0(\text{proj } \Lambda)$ of $\text{proj } \Lambda$, then we call $(g_i)_{i \in Q_0} \in \mathbb{Z}^{Q_0}$ the **g-vector** of X and denote it by g^X .

Theorem 2.13. [AIR, Theorem 5.5] *The map $T \rightarrow g^T$ gives an injection from the set of isomorphism classes of 2-term presilting objects to $K_0(\text{proj } \Lambda)$.*

2.3. Weak orders on Symmetric groups. Let \mathfrak{S}_{n+1} be the $(n+1)$ -th symmetric group and $s_i \in \mathfrak{S}_{n+1}$ denotes an adjacent transposition $(i, i+1)$. Then each element $w \in \mathfrak{S}_{n+1}$ can be written in the form $w = s_{i_\ell} s_{i_{\ell-1}} \cdots s_{i_1}$. If ℓ is minimum, then we call ℓ the *length* of w and denote it by $\ell(w)$. In this case, an expression $s_{i_\ell} s_{i_{\ell-1}} \cdots s_{i_1}$ of w is said to be a *reduced expression* of w . The following is well known (see [BjB, Section 1] for example).

Theorem 2.14. *Let $w = s_{i_\ell} \cdots s_{i_1}$.*

(1) *Assume that $j < \ell$ satisfies*

$$(i) \quad s_{i_\ell} \cdots s_{i_{j+1}}(i_j) > s_{i_\ell} \cdots s_{i_{j+1}}(i_j + 1).$$

Then there exists $k < j$ such that

$$(ii) \quad s := s_{i_{k-1}} \cdots s_{i_{j+1}}(i_j) < s_{i_{k-1}} \cdots s_{i_{j+1}}(i_j + 1) =: t \text{ and } s_{i_k}(s) > s_{i_k}(t).$$

Moreover, we have

$$w = s_{i_\ell} \cdots \widehat{s_{i_k}} \cdots \widehat{s_{i_j}} \cdots s_{i_1}.$$

(2) *The inversion number of w is coincides to $\ell(w)$.*

(3) (Matsumoto's exchange condition). *If $s_{i_\ell} \cdots s_{i_1}$ is a non-reduced expression, then there exists $j < \ell$ satisfying (i) of above and so*

$$w = s_{i_\ell} \cdots \widehat{s_{i_k}} \cdots \widehat{s_{i_j}} \cdots s_{i_1}.$$

We give a proof for reader's convenience.

Proof. For $w \in \mathfrak{S}_{n+1}$, we denote by $\gamma(w)$ the inversion number of w . It is well known that

- $\gamma(s_i w) = \gamma(w) + 1 \Leftrightarrow w^{-1}(i) < w^{-1}(i+1)$.
- $\gamma(s_i w) = \gamma(w) - 1 \Leftrightarrow w^{-1}(i) > w^{-1}(i+1)$.
- $\gamma(ws_i) = \gamma(w) + 1 \Leftrightarrow w(i) < w(i+1)$.
- $\gamma(ws_i) = \gamma(w) - 1 \Leftrightarrow w(i) > w(i+1)$.

We show (1). Assume that $j < \ell$ satisfies (i). Then (ii) follows from (i) and $i_j < i_j + 1$. It is easy to check that $(s, t) = (i_k, i_k + 1)$. Hence we conclude that

$$s_{i_{k-1}} \cdots s_{i_{j+1}} s_{i_j} s_{i_{j+1}} \cdots s_{i_{k-1}} = s_{i_k}.$$

In fact, we have that

$$s_{i_{k-1}} \cdots s_{i_{j+1}}(i_j) = i_k \text{ and } s_{i_{k-1}} \cdots s_{i_{j+1}}(i_j + 1) = i_k + 1.$$

Then we obtain that

$$s_{i_k} s_{i_{k-1}} \cdots s_{i_{j+1}} = s_{i_{k-1}} \cdots s_{i_{j+1}} s_{i_j}.$$

In particular, we have

$$w = s_{i_\ell} \cdots s_{i_k} \cdots s_{i_{j+1}} s_{i_j} \cdots s_{i_1} = s_{i_\ell} \cdots \widehat{s_{i_k}} \cdots \widehat{s_{i_j}} \cdots s_{i_1}.$$

Next we prove (2). Let $s_{i_\ell} \cdots s_{i_1}$ be a reduced expression of w . By (1), we have that

$$s_{i_\ell} \cdots s_{i_{j+1}}(i_j) < s_{i_\ell} \cdots s_{i_{j+1}}(i_j + 1)$$

for any j . Hence, we obtain that

$$\gamma(w) = \gamma(s_{i_\ell} \cdots s_{i_2}) + 1 = \cdots = \gamma(1) + \ell = \ell.$$

Finally, we show the assertion (3). Suppose that

$$s_{i_\ell} \cdots s_{i_{j+1}}(i_j) < s_{i_\ell} \cdots s_{i_{j+1}}(i_j + 1)$$

for any $j \in \{1, \dots, \ell - 1\}$. Then same argument used in the proof of (2) gives that

$$\ell(w) = \gamma(w) = \ell.$$

This is a contradiction. Therefore we have (i). \square

We recall definition of the (left) weak order on \mathfrak{S}_{n+1} .

Definition 2.15. Let $w, w' \in \mathfrak{S}_{n+1}$. We write $w \leq w'$ if there exists s_{i_1}, \dots, s_{i_k} such that

$$w' = s_{i_k} \cdots s_{i_1} w \text{ and } \ell(w') = \ell(w) + k.$$

It is obvious that \leq gives a partial order on \mathfrak{S}_{n+1} . We call this partial order the *left weak order* on \mathfrak{S}_{n+1} .

Clearly $(\mathfrak{S}_{n+1}, \leq)$ is a ranked poset by the length function ℓ . Moreover, $(\mathfrak{S}_{n+1}, \leq)$ has the lattice properties i.e. for any $w, w' \in \mathfrak{S}_{n+1}$, $\{\sigma \in \mathfrak{S}_{n+1} \mid \sigma \geq w, w'\}$ admits a maximum element $w \wedge w'$ and $\{\sigma \in \mathfrak{S}_{n+1} \mid \sigma \leq w, w'\}$ admits a minimum element $w \vee w'$. (see [BjB, Section 3.2]). By definition the minimum element of $(\mathfrak{S}_{n+1}, \leq)$ is the identity $1 \in \mathfrak{S}_{n+1}$ and the maximum element is the longest element $w_0 := (n+1, n, \dots, 1) \in \mathfrak{S}_{n+1}$. Then the assignment $w \mapsto ww_0$ gives a poset isomorphism

$$(\mathfrak{S}_{n+1}, \leq) \xrightarrow{\sim} (\mathfrak{S}_{n+1}, \leq).$$

For a non-empty subset $J \subset \{1, 2, \dots, n\}$, we denote by $w_0(J) \in \mathfrak{S}_{n+1}$ the longest element of $\langle s_j \mid j \in J \rangle \subset \mathfrak{S}_{n+1}$. Then we have the following.

Proposition 2.16. Let J be a non-empty subset of $\{1, \dots, n\}$.

- (1) [BjB, Lemma 3.2.3] $\bigvee_{j \in J} s_j = w_0(J)$.
- (2) $[1, w_0(J)] = \langle s_j \mid j \in J \rangle$.
- (3) [BjB, Lemma 3.24] If $w \leq s_j w$ for any $j \in J$, then we have

$$\bigvee_{j \in J} (s_j w) = w_0(J)w.$$

Proof. We prove (2). Let $w \leq w_0(J)$. Suppose that $w \notin \langle s_j \mid j \in J \rangle$. $w \leq w_0(J)$ implies that there exists a reduced expression

$$s_{i_\ell} \cdots s_{i_1}$$

of $w_0(J)$ such that $R = \{r \mid i_t \notin J\} \neq \emptyset$. We take a minimum element r of R . Since $s_{i_{r-1}} \cdots s_{i_1}, w_0(J) \in \langle s_j \mid j \in J \rangle$, we have that $w' := s_{i_\ell} \cdots s_{i_r} \in \langle s_j \mid j \in J \rangle$. Then $s_{i_\ell} \cdots s_{i_{r+1}}(i_r) = w'(i_r + 1) > w'(i_r) = s_{i_\ell} \cdots s_{i_{r+1}}(i_r + 1)$. Hence Theorem 2.14 gives that $s_{i_\ell} \cdots s_{i_r}$ is non-reduced. This is a contradiction. \square

3. MAIN RESULT

Let $\Lambda = kQ/I$ be a basic finite dimensional algebra, where I is an admissible ideal of kQ . We consider the following condition.

Condition 3.1. (a) Q° is isomorphic to the following quiver:

$$1 \longleftrightarrow 2 \longleftrightarrow 3 \longleftrightarrow \cdots \longleftrightarrow n$$

- (b) For each arrow $x : i \rightarrow j$ with $i \neq j$ in Q , $x\Lambda e_j = e_i\Lambda e_j = e_i\Lambda x$.

- (c) For any pair (i, j) of Q_0 , $w_j^i \neq 0$ in Λ , where w_j^i is the shortest path from i to j in Q .

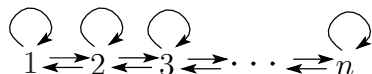
Example 3.2. (1) Let $\vec{\Delta}$ be a quiver of type A_n with linear orientation. Q denotes the double quiver of $\vec{\Delta}$ i.e. $Q_0 := \vec{\Delta}_0$ and $Q_1 := \vec{\Delta}_1 \sqcup \{\alpha^* : t(\alpha) \rightarrow s(\alpha) \mid \alpha \in \vec{\Delta}_1\}$. Then $\Pi_n := kQ/(\sum_{\alpha \in \vec{\Delta}_1} \alpha\alpha^* - \alpha^*\alpha)$ is said to be the *preprojective algebra* of type A_n .

We can easily check that the preprojective algebra Π_n of type A_n satisfies the Condition 3.1. In fact, (a) and (c) of the Condition 3.1 obviously hold. Let α be an arrow from x to y . Then the relation $\sum_{\alpha \in \vec{\Delta}_1} (\alpha\alpha^* - \alpha^*\alpha)$ induces that for any path w from x to y on Q , there exists N such that

$$w = (\alpha\alpha^*)^N \alpha = \alpha(\alpha^*\alpha)^N.$$

This gives (b) of the Condition 3.1.

- (2) The Auslander algebra of the truncated polynomial ring $k[X]/(X^n)$ satisfies the Condition 3.1.
 (3) Let Q be the following quiver:



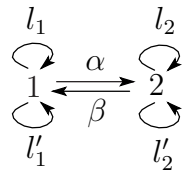
I_m denotes an admissible ideal of kQ generated by

$$\{l_i\alpha_i, l_{i+1}\alpha_i^*, \alpha_i l_{i+1}, \alpha_i^* l_i, l_i^m, \alpha_i \alpha_i^*, \alpha_i^* \alpha_i \mid i \in Q_0\},$$

where α_i (resp. α_i^*) is the arrow from i to $i+1$ (resp. from $i+1$ to i) and l_i is the loop on i (in the case that $m = 1$, we regard $Q = Q^\circ$ and I_1 is generated by $\{\alpha_i \alpha_i^*, \alpha_i^* \alpha_i \mid i \in Q_0\}$). Then $\Lambda_m := kQ/I_m$ satisfies the Condition 3.1.

We remark that for any algebra Λ satisfying the Condition 3.1, there is a surjective algebra homomorphism from Λ to Λ_1 .

- (4) Let Q be the following quiver.



Let I be an admissible ideal of kQ generated by

$$\{(\alpha\beta)^2, (\beta\alpha)^2, l_1\alpha - \alpha(l_2 + l'_2), l'_1\alpha - \alpha l'_2, l_2\beta - \beta l'_1, l'_2\beta - \beta l_1, l_i^2, l_i'^2, l_i l'_i, l'_i l_i \mid i = 1, 2\}$$

Then $\Gamma = kQ/I$ satisfies the Condition 3.1.

Main result of this paper is the following.

Theorem 3.3. *Let $\Lambda = kQ/I$ is a finite dimensional algebra with I being an admissible ideal of kQ . Then $\text{st-tilt } \Lambda \simeq (\mathfrak{S}_{n+1}, \leq)$ if and only if Λ satisfies the Condition 3.1.*

Remark 3.4. By using [EJR, Theorem 4.1] (and Theorem 1.1), we can construct infinitely many algebras whose support τ -tilting posets are isomorphic to $(\mathfrak{S}_{n+1}, \leq)$. In fact, let Λ_m be the algebra considered in Example 3.2 (3). Then $z_m := l_1^{m-1} + l_2^{m-1} + \cdots + l_n^{m-1}$ is in $\text{Rad } \Lambda \cap Z(\Lambda_m)$, where $Z(\Lambda_m)$ is the center of Λ_m . It is easy to check that $\Lambda_m/(z_m) = \Lambda_{m-1}$.

Hence [EJR, Theorem 4.1] implies that $\text{s}\tau\text{-tilt } \Lambda_m \simeq \text{s}\tau\text{-tilt } \Lambda_1$ for any $m \geq 1$. Also by using [EJR, Theorem 4.1] and Theorem 1.1, we have that

$$\text{s}\tau\text{-tilt } \Lambda_m \simeq \text{s}\tau\text{-tilt } \Lambda_1 \simeq \text{s}\tau\text{-tilt } \Pi_n \simeq (\mathfrak{S}_{n+1}, \leq).$$

Therefore [EJR, Theorem 4.1] is very powerful. But unfortunately, there exists an algebra Λ with $\text{s}\tau\text{-tilt } \Lambda \simeq (\mathfrak{S}_{n+1}, \leq)$ such that we can't prove $\text{s}\tau\text{-tilt } \Lambda \simeq (\mathfrak{S}_{n+1}, \leq)$ by using [EJR, Theorem 4.1] and Theorem 1.1.

We denote by $\Lambda \rightsquigarrow \Lambda'$ if there exists $z \in \text{Rad } \Lambda \cap Z(\Lambda)$ such that $\Lambda/(z) \cong \Lambda'$. Let \sim be the equivalence relation on the set of (isomorphism classes of) basic finite dimensional algebras generated by \rightsquigarrow . Then we can show that

$$\Gamma \not\sim \Pi_2,$$

where Γ is the algebra considered in Example 3.2 (4). Note that if $\Lambda \rightsquigarrow \Lambda^{(1)}, \Lambda^{(2)}$, then there is Λ' such that $\Lambda^{(1)}, \Lambda^{(2)} \rightsquigarrow \Lambda'$. In particular, $\Lambda^{(1)} \sim \Lambda^{(2)}$ if and only if there exists Λ such that

$$\Lambda^{(1)} \rightsquigarrow \dots \rightsquigarrow \Lambda \text{ and } \Lambda^{(2)} \rightsquigarrow \dots \rightsquigarrow \Lambda.$$

Now suppose that $\Gamma \sim \Pi_2(\sim \Lambda_1)$. Since Λ_1 has no non-zero element in $\text{Rad } \Lambda_1 \cap Z(\Lambda_1)$, there is a path

$$\Gamma \rightsquigarrow \dots \rightsquigarrow \Lambda_1.$$

Let $z \in \text{Rad } \Gamma \cap Z(\Gamma)$. Since $\text{Rad } \Gamma \cap Z(\Gamma) \subset e_1(\text{Rad } \Gamma)e_1 + e_2(\text{Rad } \Gamma)e_2$ and

$$l_1, l'_1\alpha\beta, l_1\alpha\beta, l'_1\alpha\beta, l_2, l'_2\beta\alpha, l_2\beta\alpha, l'_2\beta\alpha$$

form a basis of $e_1(\text{Rad } \Gamma)e_1 + e_2(\text{Rad } \Gamma)e_2$, we can write

$$z = al_1 + bl'_1 + c\alpha\beta + dl_1\alpha\beta + el'_1\alpha\beta + a'l_2 + b'l'_2 + c'\beta\alpha + d'l_2\beta\alpha + e'l'_2\beta\alpha.$$

By $z\alpha = \alpha z$ and $z\beta = \beta z$, we obtain that $a = a' = b = b' = d = d' = e = e' = 0$ and $c = c'$. Then $c = c' = 0$ follows from $l_1z = zl_1$. This implies that $\text{Rad } \Gamma \cap Z(\Gamma) = 0$. Therefore we have that $\Gamma \simeq \Lambda_1$ and reach a contradiction. $\Gamma \not\sim \Pi_2$ says that we can't prove $\text{s}\tau\text{-tilt } \Gamma \simeq (\mathfrak{S}_3, \leq)$ by using [EJR, Theorem 4.1] and Theorem 1.1.

4. PROOF OF THEOREM 3.3

In this section, we give a proof of Theorem 3.3. From now on, we put $P_i = e_i\Lambda$ the indecomposable projective module of Λ associated with $i \in Q_0$. Also we put $X_i := e_i\Lambda/e_i\Lambda(1-e_i)\Lambda \simeq \Lambda/(1-e_i)$. Note that X_i is in $\text{s}\tau\text{-tilt } \Lambda$ with $\text{Supp}(X_i) = \{i\}$. Therefore we have that $\text{dp}(0) = \{X_i \mid i \in Q_0\}$.

4.1. Case $n = 2$. In this subsection, we see that Theorem 3.3 hold for the case that $n = 2$. The following results are proved in [AK]. We give proofs for reader's convenience.

Lemma 4.1. [AK] *Let Q be a quiver with precisely two vertices, say 1, 2, and I an admissible ideal of kQ . Suppose that there is an arrow α from 1 to 2. Put $\Lambda := kQ/I$. Then $X_1 \oplus P_2$ is not τ -rigid. Moreover, we have $\text{Hom}_\Lambda(P_1, \tau X_1) = 0$ if and only if α is a unique arrow from 1 to 2 and $\alpha\Lambda e_2 = e_1\Lambda e_2 = e_1\Lambda\alpha$.*

Proof. Let $P_{X_1} = [P_2^r \xrightarrow{d} P_1]$ be a two term presilting complex associated with X_1 . We can easily see that $\text{Hom}_\Lambda(P_2, \tau X_1) \neq 0$. In fact, we have that $\text{Hom}_{\mathbf{K}^b(\text{proj } \Lambda)}(P_{X_1}, P_2[1]) \neq 0$.

We assume that $\text{Hom}_{\mathbf{K}^b(\text{proj } \Lambda)}(P_{X_1}, P_1[1]) = 0 (\Leftrightarrow \text{Hom}_\Lambda(P_1, \tau X_1) = 0)$. Then any homomorphism $f \in \text{Hom}_\Lambda(P_2^r, P_1)$ factors through d . By considering $f = \overbrace{(\alpha, 0, \dots, 0)}^r : P_2^r \rightarrow P_1$, we obtain that $r = 1$. In particular, we may regard d as an element of $e_1 \Lambda e_2$ and get that

$$(e_1 \Lambda e_1) d = e_1 \Lambda e_2.$$

Hence there exists $x \in e_1 \Lambda e_1 \setminus e_1(\text{Rad } \Lambda)e_1$ such that $\alpha = xd$. We conclude that

$$e_1 \Lambda \alpha = e_1 \Lambda x d = e_1 \Lambda d = e_1 \Lambda e_2.$$

Note that $\text{Im } d = d\Lambda = e_1 \Lambda e_2 \Lambda$. Therefore there also exists $y \in e_2 \Lambda e_2 \setminus e_2(\text{Rad } \Lambda)e_2$ such that $\alpha = dy$. This implies that

$$\alpha \Lambda = dy \Lambda = d \Lambda = e_1 \Lambda e_2 \Lambda.$$

Hence, α is a unique arrow from 1 to 2 and we have

$$\alpha \Lambda e_2 = e_1 \Lambda e_2 = e_1 \Lambda \alpha.$$

If α is a unique arrow from 1 to 2 and $\alpha \Lambda e_2 = e_1 \Lambda e_2 = e_1 \Lambda \alpha$ holds, then it is easy to check that

$$P_{X_1} = [P_2 \xrightarrow{\alpha} P_1]$$

gives a minimal projective presentation of X_1 and $\text{Hom}_{\mathbf{K}^b(\text{proj } \Lambda)}(P_{X_1}, P_1[1]) = 0$. \square

The following proposition is an immediate consequence of Lemma 4.1.

Proposition 4.2. *Let Q and Λ be as in Lemma 4.1.*

$\text{st-tilt } \Lambda \simeq (\mathfrak{S}_3, \leq)$ if and only if the following hold.

- (a) *There exists a unique arrow α of Q from 1 to 2 and $\alpha \Lambda e_2 = e_1 \Lambda e_2 = e_1 \Lambda \alpha$.*
- (b) *There exists a unique arrow β of Q from 2 to 1 and $\beta \Lambda e_1 = e_2 \Lambda e_1 = e_2 \Lambda \beta$.*

Proof. Since $\text{dp}(0) = \{X_1, X_2\}$, $\text{st-tilt } \Lambda \simeq (\mathfrak{S}_3, \leq)$ if and only if there are a path from Λ to X_1 with length 2 and a path from Λ to X_2 with length 2. This is equivalent to that X_1, X_2 are not projective and $X_1 \oplus P_i, X_2 \oplus P_j$ are τ -rigid with $\{i, j\} = \{1, 2\}$. Then the assertion follows from Lemma 4.1. \square

4.2. ‘if’ part. We assume that Λ satisfies the Condition 3.1.

For vertices $i, j \in Q_0$, we denote by f_i^j the homomorphism from P_i to P_j given by the path w_i^j i.e. $f_i^j(e_i \lambda) = w_i^j \lambda \in P_j$.

Lemma 4.3. (1) *Let $\alpha : i \rightarrow j$ be an arrow of Q° . Then we have*

$$e_i(\text{Rad } \Lambda) \alpha = \alpha(\text{Rad } \Lambda) e_j$$

- (2) *Let $w \in e_i \Lambda e_j$. Then there is a unique $a \in k$ and (not necessary unique) $l \in e_i(\text{Rad } \Lambda) e_i$ such that $w = (a + l) w_j^i$ in Λ . Also there is a unique $a' \in k$ and (not necessary unique) $l' \in e_j(\text{Rad } \Lambda) e_j$ such that $w = w_j^i(a' + l')$ in Λ . Furthermore, we have $a = a'$.*
- (3) *Let $f \in \text{Hom}_\Lambda(P_j, P_i)$ and $V \subset Q_0$. Assume that f is given by $w = (a e_i + l) w_j^i$ with $a \neq 0$. Then f factors through $P = \bigoplus_{t \in V} P_t$ if and only if w_j^i factors through some $t \in V$.*

Proof. We show (1). Let $l \in e_i(\text{Rad } \Lambda)e_i$. Then

$$l\alpha \in e_i\Lambda e_j = \alpha\Lambda e_j.$$

Accordingly, there exists $a \in k$ and $l' \in e_j(\text{Rad } \Lambda)e_j$ such that

$$l\alpha = \alpha(a + l').$$

If $a \neq 0$, then $a + l'$ is an invertible element in $\Lambda = kQ/I$. Therefore we have that $\alpha - l\alpha l'' \equiv 0 \pmod{I}$ for some $l'' \in kQ$. Since I is admissible, we reach a contradiction. Hence we conclude that

$$e_i(\text{Rad } \Lambda)\alpha = e_i(\text{Rad } \Lambda)e_i\alpha \subset \alpha(\text{Rad } \Lambda)e_j.$$

Similarly, one can check that

$$e_i(\text{Rad } \Lambda)e_i\alpha \supset \alpha e_j(\text{Rad } \Lambda)e_j.$$

Next we prove (2). Existence of $a \in k$ and $l \in e_i(\text{Rad } \Lambda)e_i$ directly follows from Condition 3.1 (b). Suppose that

$$w = (a_1 + l_1)w_j^i = (a_2 + l_2)w_j^i.$$

It is sufficient to show that $a_1 = a_2$. If $a_1 \neq a_2$, then $(a_1 - a_2 + l_1 - l_2)w_j^i = 0$ and $a_1 - a_2 + l_1 - l_2$ is invertible. In particular, we obtain $w_j^i = 0$. This contradicts to Condition 3.1 (c). Same argument gives that there are unique $a' \in k$ and $l' \in e_j(\text{Rad } \Lambda)e_j$ such that $w = w_j^i(a' + l')$ in Λ . Then $a = a'$ follows from (1). The assertion (3) follows from (1) and (2). \square

Theorem 4.4. [M] *Let Λ be the preprojective algebra of type A_n . For $i \in Q_0$, we let $I_i = (1 - e_i)$.*

(1) (See also [BIRS, III]). *Let $w \in \mathfrak{S}_{n+1}$. If $s_{i_\ell} \cdots s_{i_1}$ and $s_{j_\ell} \cdots s_{j_1}$ are reduced expression of w , then we have*

$$I_{i_\ell} \cdots I_{i_1} = I_{j_\ell} \cdots I_{j_1}.$$

In this case, we denote $I_{i_\ell} \cdots I_{i_1}$ by I_w .

(2) $I_w \in \text{s}\tau\text{-tilt } \Lambda$.

(3) *The map $w \mapsto I_{w w_0}$ gives a poset isomorphism*

$$(\mathfrak{S}_{n+1}, \leq) \xrightarrow{\sim} \text{s}\tau\text{-tilt } \Lambda.$$

We let $\Xi := \{\mathbf{i} = \{i_0 < i_1 < \cdots < i_{2m}\} \subset \{0, 1, \dots, n+1\} \mid m \geq 0, I \neq \{0\}, \{n+1\}\}$. For $\Xi \ni \mathbf{i} = \{i_0 < i_1 < \cdots < i_{2m}\}$, we set $m_{\mathbf{i}} := m$.

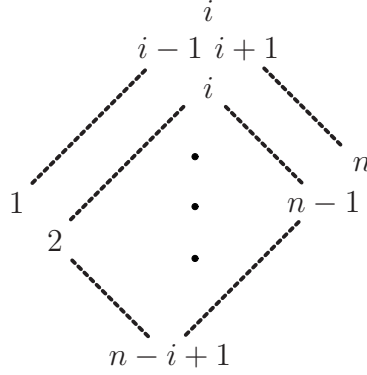
Lemma 4.5. *If $\Lambda = kQ/I$ is a preprojective algebra of type A_n , then*

$$\#\tau\text{-rigid } \Lambda \leq \#\{\mathbf{i} \in \Xi \mid m_{\mathbf{i}} > 0\}.$$

Proof. Since any indecomposable τ -rigid module X is in $\text{add } I_w$ for some $w \in \mathfrak{S}_{n+1}$ and $I_w = e_1 I_w \oplus \cdots \oplus e_n I_w$, one sees that

$$\tau\text{-rigid } \Lambda = \{e_i I_w \mid 1 \leq i \leq n, w \in \mathfrak{S}_{n+1}\} \setminus \{0\}.$$

Note that $e_i\Lambda$ has following form (loewy series).



Let $w \in \mathfrak{S}_{n+1}$ and $j \in \{1, \dots, n\}$. We show that $e_i I_w / e_i I_w (1 - e_j) \Lambda$ is either a simple module or 0. Assume that

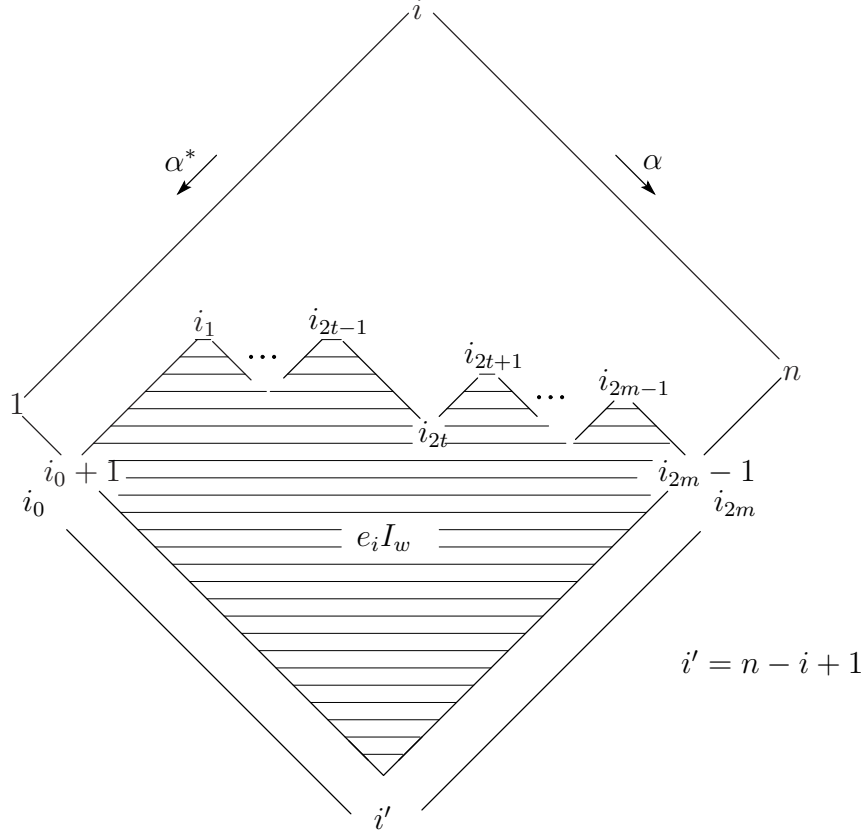
$$e_i I_w / e_i I_w (1 - e_j) \Lambda \neq 0.$$

Let $\lambda = e_i \lambda \in e_i I_w$ such that $\bar{\lambda} := (\lambda + e_i I_w (1 - e_j) \Lambda) \neq 0$. Since $\bar{\lambda}(1 - e_j) = 0$, we may assume that $\lambda \in e_i I_w e_j$. By the relation of Λ , one sees that

$$\lambda = w_j^i(a + (\alpha_j \alpha_j^*)f(\alpha_j \alpha_j^*))(\alpha_j \alpha_j^*)^N,$$

for some $N \in \mathbb{Z}_{\geq 0}$, $a \in k \setminus \{0\}$ and $f(X) \in k[X]$. Hence we also may assume that $\lambda = w_j^i(\alpha_j \alpha_j^*)^N$. Now we let $\lambda' \in e_i I_w$. Above argument implies that there are $N' \in \mathbb{Z}_{\geq 0}$, $b \in k \setminus \{0\}$ and $g(X) \in k[X]$ such that $\lambda' = w_j^i(b + (\alpha_j \alpha_j^*)g(\alpha_j \alpha_j^*))(\alpha_j \alpha_j^*)^{N'}$. Then $N > N'$ implies that $\bar{\lambda} = \overline{w_j^i(\alpha_j \alpha_j^*)^N} = 0$. Therefore one has that $N \leq N'$. In particular, $\bar{\lambda}' \in \bar{\lambda} \Lambda$. Thus $e_i I_w / e_i I_w (1 - e_j) \Lambda$ is a simple module associated with j .

Hence one obtains that $e_i I_w$ has following form.



Accordingly, $e_i I_w$ determines a polygon \mathcal{P} whose vertices are $v_0, v_1, v_2, \dots, v_{2m-1}, v_{2m}$ and v corresponding to $i_0, i_1, i_2, \dots, i_{2m-1}, i_{2m}$ and $n - i + 1$. We note that $i_0 \neq i_{2m}$. Now we input \mathcal{P} in \mathbb{R}^2 by following correspondence:

- $v_0 = (0, 0)$.
- $\alpha = (1, -1)$.
- $\alpha^* = (-1, -1)$.

Then we have

$$v_{2t-1} = (i_{2t-1} - i_0, i_{2t-1} - i_0 + 2 \sum_{s=1}^{t-1} i_{2s-1} - 2 \sum_{s=1}^{t-1} i_{2s}),$$

$$v_{2t} = (i_{2t} - i_0, -i_{2t} - i_0 + 2 \sum_{s=1}^t i_{2s-1} - 2 \sum_{s=1}^{t-1} i_{2s}).$$

Let $v_{2m} = (i_{2m} - i_0, -i_{2m} - i_0 + 2 \sum_{s=1}^m i_{2s-1} - 2 \sum_{s=1}^{m-1} i_{2s}) =: (x, y)$. Since v is the intersection of $\{a(1, -1) \mid a \in \mathbb{R}\}$ and $\{b(1, 1) + v_{2m} \mid b \in \mathbb{R}\}$, we conclude that

$$(n - i - i_0 + 1, -(n - i - i_0 + 1)) = v = \left(\frac{x - y}{2}, \frac{-x + y}{2}\right).$$

Therefore $e_i I_w$ is uniquely determined by $\mathbf{i} = (i_0 < i_1 < \dots < i_{2m}) \in \Xi$. In particular, τ -rigid Λ is parametrized by a subset of

$$\{\mathbf{i} \in \Xi \mid m_{\mathbf{i}} > 0\}.$$

□

For $\mathbf{i} \in \Xi$, we set a two-term objects $X_{\mathbf{i}}(\Lambda) = X_{\mathbf{i}} = [X_{\mathbf{i}}^{-1} \xrightarrow{d_{\mathbf{i}}} X_{\mathbf{i}}^0]$ as follows:

$$\begin{array}{ccc}
 (-1\text{th}) & & (0\text{th}) \\
 & & \\
 P_{i_0} & \xrightarrow{f_{i_0}^{i_1}} & P_{i_1} \\
 & \nearrow f_{i_2}^{i_1} & \nearrow \\
 P_{i_2} & & \\
 & \vdots & \\
 & \vdots & \\
 & \vdots & \\
 P_{i_{2m_{\mathbf{i}}-2}} & \xrightarrow{f_{i_{2m_{\mathbf{i}}-2}}^{i_{2m_{\mathbf{i}}-3}}} & P_{i_{2m_{\mathbf{i}}-3}} \\
 & \searrow f_{i_{2m_{\mathbf{i}}-2}}^{i_{2m_{\mathbf{i}}-1}} & \searrow \\
 & & P_{i_{2m_{\mathbf{i}}-1}} \\
 P_{i_{2m_{\mathbf{i}}}} & \xrightarrow{f_{i_{2m_{\mathbf{i}}}}^{i_{2m_{\mathbf{i}}-1}}} &
 \end{array}$$

where we assume $P_0 = P_{n+1} = 0$.

Lemma 4.6. $X_{\mathbf{i}}$ is indecomposable.

Proof. It is sufficient to show that $\text{End}_{\mathcal{K}^b(\text{proj } \Lambda)}(X_{\mathbf{i}})$ is local. Let $\varphi = (u, v) \in \text{End}_{\mathcal{K}^b(\text{proj } \Lambda)}(X_{\mathbf{i}})$, where $u \in \text{End}_{\Lambda}(\bigoplus_t P_{i_{2t}})$ and $v \in \text{End}_{\Lambda}(\bigoplus_t P_{i_{2t+1}})$. Denote by $u_t^s : P_{i_{2t}} \rightarrow P_{i_{2s}}$ and $v_t^s : P_{i_{2t+1}} \rightarrow P_{i_{2s+1}}$ given by u and v respectively. If $t \neq s$, then u_t^s and v_t^s are in radical of $\text{mod } \Lambda$. Hence u (resp. v) is an isomorphism if and only if u_t^t (resp. v_t^t) is isomorphism for any t .

By Lemma 4.3 (2), we can easily check that if u_t^t (resp. v_t^t) is an isomorphism, then v_t^t and v_{t-1}^{t-1} (resp. u_t^t and u_{t+1}^{t+1}) are isomorphisms. In this case, we have that φ is an isomorphism. Therefore, φ is not an isomorphism only if u_t^s and v_t^s are in radical of $\text{mod } \Lambda$ for any t, s . Conversely, if u_t^s and v_t^s are in radical of $\text{mod } \Lambda$ for any t, s , then φ is not an isomorphism. In particular, the set of non-isomorphisms of $\text{End}_{\mathcal{K}^b(\text{proj } \Lambda)}(X_{\mathbf{i}})$ form an ideal. This gives the assertion. \square

Lemma 4.7. Let $\mathbf{i}, \mathbf{j} \in \Xi$. For a pair (t, s) such that $0 < i_{2t} < j_{2s+1} < n+1$, we define two sequences $\mathbf{t}^+(t, s) := (t = t_0 \leq t_1 \leq t_2 \leq \dots)$ and $\mathbf{s}^+(t, s) := (s = s_0 \leq s_1 \leq s_2 \leq \dots)$ by following rule:

$$(i) \quad t_r := \begin{cases} \max\{t \leq m_{\mathbf{i}} + 1 \mid i_{2t-2} < j_{2s_{r-1}+1}\} & \text{if } s_{r-1} \leq m_{\mathbf{j}} - 1 \\ m_{\mathbf{i}} + 1 & \text{if } s_{r-1} \geq m_{\mathbf{j}}, t_{r-1} \leq m_{\mathbf{i}} \\ m_{\mathbf{i}} + 2 & \text{if } t_{r-1} \geq m_{\mathbf{i}} + 1 \end{cases}$$

$$(ii) \quad s_r := \begin{cases} \max\{s \leq m_j \mid j_{2s-1} < i_{2t_r}\} & \text{if } t_r \leq m_i \\ m_j & \text{if } s_{r-1} < m_j, t_r = m_i + 1 \\ m_j + 1 & \text{if } s_{r-1} \geq m_j. \end{cases}$$

Also we define two sequences $\mathbf{t}^-(t, s) := (t = t_0 \geq t_{-1} \geq t_{-2} \geq \cdots)$ and $\mathbf{s}^-(t, s) := (s = s_0 \geq s_{-1} \geq s_{-2} \geq \cdots)$ by following rule:

$$(iii) \quad s_r := \begin{cases} \min\{s \geq -1 \mid j_{2s+3} > i_{2t_{r+1}}\} & \text{if } t_{r+1} \geq 0 \\ -1 & \text{if } t_{r+1} \leq -1, s_{r+1} \geq 0 \\ -2 & \text{if } s_{r+1} \leq -1. \end{cases}$$

$$(iv) \quad t_r := \begin{cases} \min\{t \geq -1 \mid i_{2t+2} > j_{2s_{r+1}}\} & \text{if } s_r \geq 0 \\ -1 & \text{if } s_r = -1, t_{r+1} \geq 0 \\ -2 & \text{if } t_{r+1} \leq -1 \end{cases}$$

If $\text{Hom}_{\mathbf{K}^b(\text{proj } \Lambda)}(X_i, X_j[1]) = 0$, then one of the following holds.

- (1) We have (a), (b), (c) and (d).
 - (a) $i_{2t_r} < i_{2t_{r+1}} \leq j_{2s_{r+1}}$ for any $r \geq 0$ such that $t_r \leq m_i$.
 - (b) $i_{2t_{r-2}} < i_{2t_{r-1}} \leq j_{2s_{r-1}+1}$ for any $r \geq 1$ such that $t_r \leq m_i + 1$.
 - (c) $j_{2s_{r-1}+1} < j_{2s_{r-1}+2} \leq i_{2t_r}$ for any $r \geq 1$ such that $s_r \leq m_j$.
 - (d) $j_{2s_{r-1}} < j_{2s_r} \leq i_{2t_r}$ for any $r \geq 1$ such that $s_r \leq m_j$.
 Where we put $i_{2m_i+1} = j_{2m_j+1} = n + 2$ and $i_{2m_i+1} = j_{2m_j+1} = n + 3$.
- (2) We have (a'), (b'), (c') and (d').
 - (a') $j_{2s_{r+1}} > j_{2s_r} \geq i_{2t_r}$ for any $r \leq 0$ such that $s_r \geq 0$.
 - (b') $j_{2s_{r+3}} > j_{2s_{r+2}} \geq i_{2t_{r+1}}$ for any $r \leq -1$ such that $s_r \geq -1$.
 - (c') $i_{2t_{r+1}} > i_{2t_{r+1}-1} \geq j_{2s_{r+1}}$ for any $r \leq -1$ such that $t_r \geq -1$.
 - (d') $i_{2t_{r+2}} > i_{2t_{r+1}} \geq j_{2s_{r+1}}$ for any $r \leq -1$ such that $t_r \geq -1$.
 Where we put $i_{-1} = j_{-1} = -1$ and $i_{-2} = j_{-2} = -2$.

Proof. We note that if $t_r \leq m_i$ (resp. $s_r \leq m_j$) for $r \geq 0$, then $t_r < t_{r+1}$ (resp. $s_r < s_{r+1}$). and if $t_r \geq -1$ (resp. $s_r \geq -1$) for $r \leq 0$, then $t_r > t_{r-1}$ (resp. $s_r > s_{r-1}$). We also note that $\text{Hom}_{\mathbf{K}^b(\text{proj } \Lambda)}(X_i, X_j[1]) = 0$ implies that $i_{2p} \neq j_{2q+1}$ for any p, q . We first show the assertion in the case that $t_1 = t_0 + 1$. Let

$$\begin{aligned} \xi^+ &:= \{(i_{2t_0+1}, j_{2s_0+1}) \mid i_{2t_0+1} > j_{2s_0+1}\} (= \xi_0^+) \\ &\sqcup \{(i_{2t_{r-1}}, j_{2s_{r-1}+1}) \mid r \geq 1, i_{2t_{r-1}} > j_{2s_{r-1}+1}, t_r \leq m_i\} (= \xi_1^+) \\ &\sqcup \{(i_{2t_r}, j_{2s_{r-1}+2}) \mid r \geq 1, i_{2t_r} < j_{2s_{r-1}+2}, s_r \leq m_j\} (= \xi_2^+) \\ &\sqcup \{(i_{2t_r}, j_{2s_r}) \mid r \geq 1, i_{2t_r} < j_{2s_r}, s_r \leq m_j\} (= \xi_3^+) \\ &\sqcup \{(i_{2t_{r+1}}, j_{2s_{r+1}}) \mid r \geq 1, i_{2t_{r+1}} > j_{2s_{r+1}}, t_r \leq m_i\} (= \xi_4^+). \\ \xi^- &:= \{(i_{2t_0}, j_{2s_0}) \mid i_{2t_0} > j_{2s_0}\} (= \xi_0^-) \\ &\sqcup \{(i_{2t_{r+1}}, j_{2s_{r+2}}) \mid r \leq -1, i_{2t_{r+1}} > j_{2s_{r+2}}, s_r \geq -1\} (= \xi_1^-) \\ &\sqcup \{(i_{2t_{r+1}-1}, j_{2s_{r+1}}) \mid r \leq -1, i_{2t_{r+1}-1} < j_{2s_{r+1}}, t_r \geq -1\} (= \xi_2^-) \\ &\sqcup \{(i_{2t_{r+1}}, j_{2s_{r+1}}) \mid r \leq -1, i_{2t_{r+1}} < j_{2s_{r+1}}, t_r \geq -1\} (= \xi_3^-) \\ &\sqcup \{(i_{2t_r}, j_{2s_r}) \mid r \leq -1, i_{2t_r} > j_{2s_r}, s_r \geq 0\} (= \xi_4^-). \end{aligned}$$

It is sufficient to show that either $\xi^+ = \emptyset$ or $\xi^- = \emptyset$ holds. We assume that $\xi^+ \neq \emptyset$. We consider lexicographical order \preceq on \mathbb{Z}^2 (i.e. $(x, y) \preceq (x', y')$ if either (i) $x < x'$, or (ii) $x = x'$, $y \leq y'$ hold) and take a minimum element $(x, y) \in \xi^+$.

Claim 1. Suppose that $\xi^- \neq \emptyset$. Let (x', y') be a maximum element of ξ^- .

(1) Assume that $(x', y') = (i_{2t_{r'}+1}, j_{2s_{r'}+2}) \in \xi_1^-$. If $r' = -1$, then we have

$$i_{2t_0+1} \leq j_{2s_{-1}+3}.$$

If $r' < -1$, then the following holds.

$$\left\{ \begin{array}{l} & & & j_{2s_{r'}+3} = j_{2s_{r'}+1+1} = i_{2t_{r'}+1+1} < \cdots \\ < i_{2t_p+2} = i_{2t_{p+1}} = j_{2s_p+2} < j_{2s_p+3} = j_{2s_{p+1}+1} = i_{2t_{p+1}+1} < \cdots \\ < i_{2t_{-2}+2} = i_{2t_{-1}} = j_{2s_{-2}+2} < j_{2s_{-2}+3} = j_{2s_{-1}+1} = i_{2t_{-1}+1} \\ < i_{2t_{-1}+2} = i_{2t_0} = j_{2s_{-1}+2} < j_{2s_{-1}+3} \geq i_{2t_0+1} \end{array} \right.$$

(2) Assume that $(x', y') = (i_{2t_{r'}+1-1}, j_{2s_{r'}+1}) \in \xi_2^-$. If $r' = -1$, then we have

$$i_{2t_0} = j_{2s_{-1}+2} < j_{2s_{-1}+3} \geq i_{2t_0+1}.$$

If $r' < -1$, then the following holds.

$$\left\{ \begin{array}{l} & i_{2t_{r'}+1} = j_{2s_{r'}+2} < j_{2s_{r'}+3} = j_{2s_{r'}+1+1} = i_{2t_{r'}+1+1} < \cdots \\ < i_{2t_p+2} = i_{2t_{p+1}} = j_{2s_p+2} < j_{2s_p+3} = j_{2s_{p+1}+1} = i_{2t_{p+1}+1} < \cdots \\ < i_{2t_{-2}+2} = i_{2t_{-1}} = j_{2s_{-2}+2} < j_{2s_{-2}+3} = j_{2s_{-1}+1} = i_{2t_{-1}+1} \\ < i_{2t_{-1}+2} = i_{2t_0} = j_{2s_{-1}+2} < j_{2s_{-1}+3} \geq i_{2t_0+1} \end{array} \right.$$

(3) Assume that $(x', y') = (i_{2t_{r'}+1}, j_{2s_{r'}+1}) \in \xi_3^-$. Then we obtain $i_{2t_{r'}+2} > j_{2s_{r'}+1} > 0$. If $r' = -1$, then we have

$$i_{2t_{-1}+2} = i_{2t_0} = j_{2s_{-1}+2} < j_{2s_{-1}+3} \geq i_{2t_0+1}.$$

If $r' < -1$, then the following holds.

$$\left\{ \begin{array}{l} i_{2t_{r'}+2} = i_{2t_{r'}+1} = j_{2s_{r'}+2} < j_{2s_{r'}+3} = j_{2s_{r'}+1+1} = i_{2t_{r'}+1+1} < \cdots \\ < i_{2t_p+2} = i_{2t_{p+1}} = j_{2s_p+2} < j_{2s_p+3} = j_{2s_{p+1}+1} = i_{2t_{p+1}+1} < \cdots \\ < i_{2t_{-2}+2} = i_{2t_{-1}} = j_{2s_{-2}+2} < j_{2s_{-2}+3} = j_{2s_{-1}+1} = i_{2t_{-1}+1} \\ < i_{2t_{-1}+2} = i_{2t_0} = j_{2s_{-1}+2} < j_{2s_{-1}+3} \geq i_{2t_0+1} \end{array} \right.$$

(4) Assume that $(x', y') = (i_{2t_{r'}}, j_{2s_{r'}}) \in \xi_4^-$. Then we obtain $0 < i_{2t_{r'}} < i_{2t_{r'}+1} = j_{2s_{r'}+1}$. If $r' = -1$, then we have

$$i_{2t_{-1}+2} = i_{2t_0} = j_{2s_{-1}+2} < j_{2s_{-1}+3} \geq i_{2t_0+1}.$$

If $r' < -1$, then the following holds.

$$\left\{ \begin{array}{l} i_{2t_{r'}+2} = i_{2t_{r'}+1} = j_{2s_{r'}+2} < j_{2s_{r'}+3} = j_{2s_{r'}+1+1} = i_{2t_{r'}+1+1} < \cdots \\ < i_{2t_p+2} = i_{2t_{p+1}} = j_{2s_p+2} < j_{2s_p+3} = j_{2s_{p+1}+1} = i_{2t_{p+1}+1} < \cdots \\ < i_{2t_{-2}+2} = i_{2t_{-1}} = j_{2s_{-2}+2} < j_{2s_{-2}+3} = j_{2s_{-1}+1} = i_{2t_{-1}+1} \\ < i_{2t_{-1}+2} = i_{2t_0} = j_{2s_{-1}+2} < j_{2s_{-1}+3} \geq i_{2t_0+1} \end{array} \right.$$

Proof. We treat the case that $(x', y') = (i_{2t_{r'}}, j_{2s_{r'}}) \in \xi_4^-$. By definition we obtain that

$i_{2t} \geq i_{2t_{r'}+2} > j_{2s_{r'}+1} > i_{2t_{r'}} > j_{2s_{r'}} \geq 0$. Accordingly, $P_{i_{2t_{r'}}}, P_{j_{2s_{r'}+1}} \neq 0$. Let $f_{i_{2t_{r'}}}^{j_{2s_{r'}+1}} \in$

$\text{Hom}_\Lambda(P_{i_{2t_{r'}}}, P_{j_{2s_{r'}+1}})$ given by $w_{i_{2t_{r'}}}^{j_{2s_{r'}+1}}$. We consider $\varphi = (\varphi_{i_{2t}}^{j_{2s+1}} : P_{i_{2t}}^{j_{2s+1}} \rightarrow P_{j_{2s+1}}) \in \text{Hom}_\Lambda(X_i^{-1}, X_j^0) = \text{Hom}_\Lambda(\oplus P_{i_{2t}}, \oplus P_{j_{2s+1}})$ such that

$$\varphi_{i_{2t}}^{j_{2s+1}} = \begin{cases} f_{i_{2t_{r'}}}^{j_{2s_{r'}+1}} & (t, s) = (t_{r'}, s_{r'}) \\ 0 & \text{otherwise.} \end{cases}$$

We may regard φ as a morphism in $\text{Hom}_{\mathbf{K}^b(\text{proj } \Lambda)}(X_i, X_j[1])$ by natural way. Note that

$$\text{Hom}_{\mathbf{K}^b(\text{proj } \Lambda)}(X_i, X_j[1]) = 0.$$

Therefore there are $h = (h_{i_{2t+1}}^{j_{2s+1}}) \in \text{Hom}_\Lambda(X_i^0, Y_j^0)$ and $h' = (h_{i_{2t}}^{j_{2s}}) \in \text{Hom}_\Lambda(X_i^{-1}, Y_j^{-1})$ such that

$$\varphi = h \circ d_{X_i} + d_{X_j} \circ h'.$$

In particular, one sees the following equation. (See figure 1.)

$$f_{i_{2t_{r'}}}^{j_{2s_{r'}+1}} = h_{i_{2t_{r'}+1}}^{j_{2s_{r'}+1}} \circ f_{i_{2t_{r'}}}^{i_{2t_{r'}+1}} + h_{i_{2t_{r'}-1}}^{j_{2s_{r'}+1}} \circ f_{i_{2t_{r'}}}^{i_{2t_{r'}-1}} + f_{j_{2s_{r'}}}^{j_{2s_{r'}+1}} \circ h_{i_{2t_{r'}}}^{j_{2s_{r'}}} + f_{j_{2s_{r'}+2}}^{j_{2s_{r'}+1}} \circ h_{i_{2t_{r'}}}^{j_{2s_{r'}+2}}.$$

Note that $i_{2t_{r'}-1}, j_{2s_{r'}} < i_{2t_{r'}} < j_{2s_{r'}+1} < j_{2s_{r'}+2}$. Lemma 4.3 implies that

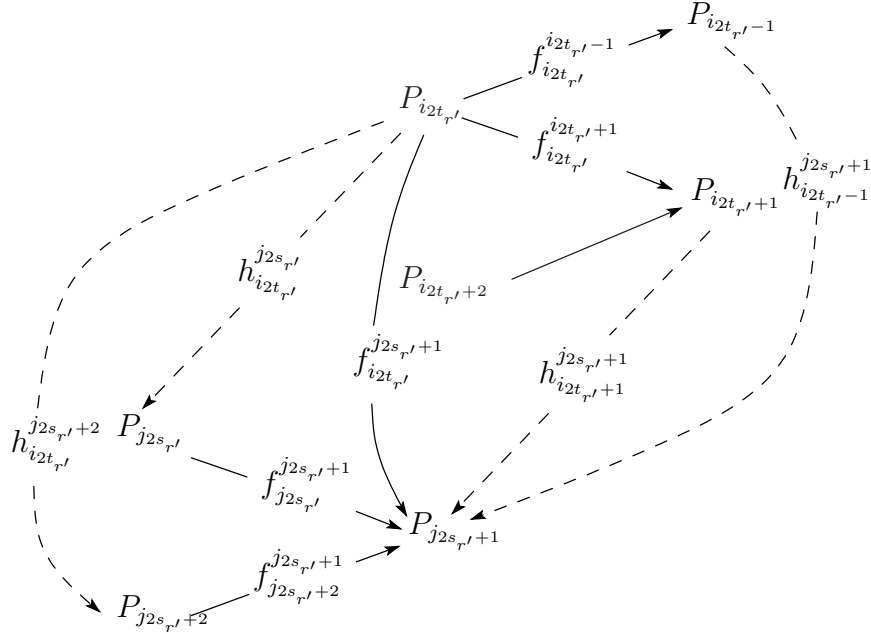


FIGURE 1.

$$i_{2t_{r'}+1} \leq j_{2s_{r'}+1}.$$

Since $(i_{2t_{r'}}, j_{2s_{r'}})$ is maximum in ξ^- and $(i_{2t_{r'}}, j_{2s_{r'}}) \prec (i_{2t_{r'}+1}, j_{2s_{r'}+1})$, we conclude that

$$j_{2s_{r'}+1} \leq i_{2t_{r'}+1}.$$

In particular, we get

$$i_{2t_{r'}+1} = j_{2s_{r'}+1}.$$

Lemma 4.3 also says that

$$h_{i_{2t_{r'}+1}}^{j_{2s_{r'}+1}} = 1 + l,$$

for some $l \in \text{Rad End}_\Lambda(P_{j_{2s_{r'}+1}})$.

Next we consider $\varphi_{i_{2t_{r'}+2}}^{j_{2s_{r'}+1}} = 0$. We obtain the following equation.

$$0 = h_{i_{2t_{r'}+1}}^{j_{2s_{r'}+1}} \circ f_{i_{2t_{r'}+2}}^{i_{2t_{r'}+1}} + h_{i_{2t_{r'}+3}}^{j_{2s_{r'}+1}} \circ f_{i_{2t_{r'}+2}}^{i_{2t_{r'}+3}} + f_{j_{2s_{r'}}}^{j_{2s_{r'}+1}} \circ h_{i_{2t_{r'}+2}}^{j_{2s_{r'}}} + f_{j_{2s_{r'}+2}}^{j_{2s_{r'}+1}} \circ h_{i_{2t_{r'}+2}}^{j_{2s_{r'}+2}}.$$

By Lemma 4.3, we have that

$$\begin{aligned} h_{i_{2t_{r'}+3}}^{j_{2s_{r'}+1}} \circ f_{i_{2t_{r'}+2}}^{i_{2t_{r'}+3}} + f_{j_{2s_{r'}}}^{j_{2s_{r'}+1}} \circ h_{i_{2t_{r'}+2}}^{j_{2s_{r'}}} + f_{j_{2s_{r'}+2}}^{j_{2s_{r'}+1}} \circ h_{i_{2t_{r'}+2}}^{j_{2s_{r'}+2}} &= (-1 + l) \circ f_{i_{2t_{r'}+2}}^{j_{2s_{r'}+1}} \\ &= f_{i_{2t_{r'}+2}}^{j_{2s_{r'}+1}} \circ (-1 + l'), \end{aligned}$$

for some $l' \in \text{Rad End}_\Lambda(P_{j_{2s_{r'}+2}})$. Then Lemma 4.3 implies that

$$j_{2s_{r'}+2} \leq i_{2t_{r'}+2}.$$

Since $(i_{2t_{r'}}, j_{2s_{r'}})$ is maximum in ξ^- and $(i_{2t_{r'}}, j_{2s_{r'}}) \prec (i_{2t_{r'}+1}, j_{2s_{r'}+2})$, we conclude that

$$i_{2t_{r'}+2} \leq i_{2t_{r'}+1} \leq j_{2s_{r'}+2}.$$

In particular, we have that

$$i_{2t_{r'}+2} = i_{2t_{r'}+1} = j_{2s_{r'}+2} \text{ and } h_{i_{2t_{r'}+1}}^{j_{2s_{r'}+2}} \equiv -1 \pmod{\text{Rad End}_\Lambda(P_{j_{2s_{r'}+2}})}.$$

If $r' < -1$, then one can check that

$$(\#) \left\{ \begin{array}{l} i_{2t_{r'}+2} = i_{2t_{r'}+1} = j_{2s_{r'}+2} < j_{2s_{r'}+3} = j_{2s_{r'}+1}+1 = i_{2t_{r'}+1}+1 < \dots \\ < i_{2t_p+2} = i_{2t_p+1} = j_{2s_p+2} < j_{2s_p+3} = j_{2s_p+1}+1 = i_{2t_p+1}+1 < \dots \\ < i_{2t_{-2}+2} = i_{2t_{-1}} = j_{2s_{-2}+2} < j_{2s_{-2}+3} = j_{2s_{-1}+1} = i_{2t_{-1}+1} \\ < i_{2t_{-1}+2} = i_{2t_0} = j_{2s_{-1}+2} < j_{2s_{-1}+3} \geq i_{2t_0+1} \end{array} \right.,$$

and

$$(\natural) \left\{ \begin{array}{l} h_{i_{2t_p+1}}^{j_{2s_p+1}} \equiv 1 \pmod{\text{Rad End}_\Lambda(P_{j_{2s_p+1}})} \\ h_{i_{2t_p+1}}^{j_{2s_p+2}} \equiv -1 \pmod{\text{Rad End}_\Lambda(P_{j_{2s_p+2}})}, \end{array} \right.$$

for any $p \in \{r', r'+1, \dots, -1\}$ inductively. In the case that $r' = -1$, we have that

$$i_{2t_0+1} \leq j_{2s_{-1}+3}.$$

In fact, we have an equation

$$f_{j_{2s_{-1}+2}}^{j_{2s_{-1}+3}} \circ h_{i_{2t_0}}^{j_{2s_{-1}+2}} + f_{j_{2s_{-1}+4}}^{j_{2s_{-1}+3}} \circ h_{i_{2t_0}}^{j_{2s_{-1}+4}} + h_{i_{2t_0-1}}^{j_{2s_{-1}+3}} \circ f_{i_{2t_0}}^{i_{2t_0-1}} + h_{i_{2t_0+1}}^{j_{2s_{-1}+3}} \circ f_{i_{2t_0}}^{i_{2t_0+1}} = 0$$

and that $h_{i_{2t_0}}^{j_{2s_{-1}+2}} \equiv -1 \pmod{\text{Rad End}_\Lambda(P_{j_{2s_{-1}+2}})}$. Accordingly, Lemma 4.3 implies the assertion. Similar argument gives remaining assertions. (We only note that if $(x', y') = (i_{2t_{r'}+1}, j_{2s_{r'}+1}) \in \xi_3^-$, then $-1 \leq i_{2t_{r'}+1} < j_{2s_{r'}+1}$ implies that $s_{r'} \geq 0$. Therefore, we obtain that $0 < j_{2s_{r'}+1} < i_{2t_{r'}+2}$. In particular, we have $P_{i_{2t_{r'}+2}} \neq 0 \neq P_{j_{2s_{r'}+1}}$.)

■

Claim 2. (1) If $(x, y) = (i_{2t_0+1}, j_{2s_0+1}) \in \xi_0^+$, then $\xi^- = \emptyset$.

(2) If $(x, y) = (i_{2t_r}, j_{2s_{r-1}+2}) \in \xi_2^+$, then we have $i_{2t_r-1} = j_{2s_{r-1}+1}$ and

$$(*) \left\{ \begin{array}{l} i_{2t_r-2} = i_{2t_r-1} = j_{2s_{r-1}} > j_{2s_{r-1}-1} = j_{2s_{r-2}+1} = i_{2t_{r-1}-1} > \dots \\ > i_{2t_p-2} = i_{2t_p-1} = j_{2s_{p-1}} > j_{2s_{p-1}-1} = j_{2s_{p-2}+1} = i_{2t_{p-1}-1} > \dots \\ > i_{2t_2-2} = i_{2t_1} = j_{2s_1} > j_{2s_1-1} = j_{2s_0+1} = i_{2t_1-1} \\ > j_{2s_0} \geq i_{2t_1-2} = i_{2t_0}. \end{array} \right.$$

(3) If $(x, y) = (i_{2t_r}, j_{2s_r}) \in \xi_3^+$, then we have $s_r = s_{r-1} + 1$ and so

$$(x, y) = (i_{2t_r}, j_{2s_r}) = (i_{2t_r}, j_{2s_{r-1}+2}).$$

(4) Assume that $(x, y) = (i_{2t_r+1}, j_{2s_r+1}) \in \xi_4^+$. If $t_r < m_i$, then $i_{2t_r} = j_{2s_r} > j_{2s_{r-1}} = j_{2s_{r-1}+1} = i_{2t_r-1}$ and $(*)$ hold. If $t_r = m_i$, then $i_{2t_r} \neq n+1$. Furthermore, we obtain that $i_{2t_r} = j_{2s_r} > j_{2s_{r-1}} = j_{2s_{r-1}+1} = i_{2t_r-1}$ and $(*)$.

(5) If $(x, y) = (i_{2t_{r+1}-1}, j_{2s_r+1}) \in \xi_1^+$ ($r \geq 0$), then we have that $t_{r+1} = t_r + 1$ and so

$$(x, y) = (i_{2t_{r+1}-1}, j_{2s_r+1}) = (i_{2t_r+1}, j_{2s_r+1}).$$

Proof. We show (1). Suppose that $\xi^- \neq \emptyset$ and take a maximum element (x', y') of ξ^- . If $(x', y') = (i_{2t_0}, j_{2s_0})$, then $i_{2t_0} > j_{2s_0}$. Now we consider $f = f_{i_{2t_0}}^{j_{2s_0}+1} \in \text{Hom}_\Lambda(P_{i_{2t_0}}, P_{j_{2s_0}+1})$ given by the path $w_{i_{2t_0}}^{j_{2s_0}+1}$. Since $\text{Hom}_{K^b(\text{proj } \Lambda)}(X_i, X_j[1]) = 0$, we have that f factors through $P_{i_{2t_0-1}} \oplus P_{i_{2t_0+1}} \oplus P_{j_{2s_0}} \oplus P_{j_{2s_0}+2}$. Note that $w_{i_{2t_0}}^{j_{2s_0}+1}$ does not factor through i_{2t_0-1} , j_{2s_0+2} and j_{2s_0} . Therefore Lemma 4.3 implies that $i_{2t_0} < i_{2t_0+1} \leq j_{2s_0+1}$. This contradicts to $i_{2t_0+1} > j_{2s_0+1}$. Hence we may assume that $(x', y') \in \xi_b^-$ ($b = 1, 2, 3, 4$). Then Claim 1 implies that

$$i_{2t_0+1} \leq j_{2s_{-1}+3} \leq j_{2s_0+1}.$$

This contradicts to the hypothesis of (1).

Next we show (2). We consider $\varphi \in \text{Hom}_\Lambda(X_i^{-1}, Y_j^0) = \text{Hom}_\Lambda(\oplus P_{i_{2p}}, \oplus P_{j_{2q+1}})$ given by $(\varphi_{i_{2p}}^{j_{2q+1}} : P_{i_{2p}} \rightarrow P_{j_{2q+1}})$ where

$$\varphi_{i_{2p}}^{j_{2q+1}} = \begin{cases} f_{i_{2t_r}}^{j_{2s_{r-1}+1}} & (p, q) = (t_r, s_{r-1}) \\ 0 & \text{otherwise.} \end{cases}$$

Then one can apply similar argument we used in the proof of Claim 1 for φ and obtain $(*)$. Likewise, we have $(*)$ in the case of (3), (4) and (5). (For the assertion (4), we remark that $(x, y) = (i_{2t_r+1}, j_{2s_r+1})$ implies that $s_r < m_j$. By definition of s_r , we have $i_{2t_r} < j_{2s_r+1} < j_{2m_j+1} = n+2$ and so $i_{2t_r} \neq n+1$.) ■

We continue a proof of Lemma 4.7. (Remark: We now assume that $\xi^+ \neq \emptyset$ and consider the case that $t_1 = t_0 + 1$.) Suppose that $\xi^- \neq \emptyset$ and take a maximum element $(x', y') \in \xi^-$. We will give a contradiction in the case that $(x, y) \in \xi_4^+$ and $(x', y') \in \xi_4^-$.

Let (φ, h, h') be a triple considered in the proof of Claim 1. Then one has (\sharp) and (\natural) . Therefore by Claim 2 (4) and (\sharp) , we conclude that

$$i_{2t_1-1} = i_{2t_0+1} \leq j_{2s_{-1}+3} \leq j_{2s_0+1} = i_{2t_1-1}.$$

In particular, we have that

$$j_{2s_{-1}+3} = j_{2s_0+1} = i_{2t_1-1}.$$

Note that $\varphi_{i_{2t_0}}^{j_{2s_0}+1} = 0$ and $h_{i_{2t_0}}^{j_{2s_0}} \equiv -1 \pmod{\text{Rad End}_\Lambda(P_{j_{2s_0}})}$ (by (4) and $s_0 = s_{-1} + 1$). Hence Lemma 4.3 and $i_{2t_1-1} = j_{2s_0+1}$ imply that

$$h_{i_{2t_1-1}}^{j_{2s_0}+1} \equiv 1 \pmod{\text{Rad End}_\Lambda(P_{j_{2s_0+1}})}.$$

Then by using Lemma 4.3 and the condition (*), one can check that

$$\begin{cases} h_{i_{2t_{p-1}}}^{j_{2s_{p-1}}+1} & \equiv 1 \pmod{\text{Rad End}_\Lambda(P_{j_{2s_{p-1}+1}})} \\ h_{i_{2t_p}}^{j_{2s_p}} & \equiv -1 \pmod{\text{Rad End}_\Lambda(P_{j_{2s_p}})}, \end{cases}$$

for any $p \in \{1, \dots, r\}$ inductively. Now Claim 2 (4) implies that $i_{2t_r} \neq n+1 (\Leftrightarrow P_{i_{2t_r}} \neq 0)$. Then $i_{2t_r+1} > j_{2s_r+1}$ gives that $s_r < m_{\mathbf{j}} (\Leftrightarrow P_{j_{2s_r+1}} \neq 0)$. Note that $\varphi_{i_{2t_r}}^{j_{2s_r}+1} = 0$ and

$$h_{i_{2t_r}}^{j_{2s_r}} \equiv -1 \pmod{\text{Rad End}_\Lambda(P_{j_{2s_r}})}.$$

If $t_r = m_{\mathbf{i}}$, then by Lemma 4.3, $w_{i_{2t_r}}^{j_{2s_r}+1}$ factors through either j_{2s_r+2} or i_{2t_r-1} . This is a contradiction. If $t_r < m_{\mathbf{i}}$, then Lemma 4.3 implies that $w_{i_{2t_r}}^{j_{2s_r}+1}$ have to through i_{2t_r+1} . In particular, we obtain that

$$i_{2t_r} < i_{2t_r+1} \leq j_{2s_r+1}.$$

This contradicts to that $(x, y) = (i_{2t_r+1}, j_{2s_r+1}) \in \xi_4^+$.

If $(x, y) \in \xi_a^+$ and $(x', y') \in \xi_b^-$, then similar argument gives a contradiction. Therefore we have the assertion in the case that $t_1 = t_0 + 1$.

Suppose that $t_0 < m_{\mathbf{i}}$ and $2t_1 - 2 > 2t$. Let $t' := t_1 - 1 > t$ and $\mathbf{t}^+(t', s) = (t'_0 < t'_1 < \dots)$, $\mathbf{s}^-(t', s) = (s'_0 < s'_1 < \dots)$, $\mathbf{t}^-(t', s) = (t'_0 > t'_{-1} > \dots)$, $\mathbf{s}^+(t', s) = (s'_0 > s'_1 > \dots)$.

By definition we have the following:

$$\begin{cases} t'_r = t_r & r \geq 1 \\ s'_r = s_r & r \geq 1 \end{cases}$$

We also obtain that $t'_1 = t'_0 + 1$. Hence the assertion holds for (t', s) . If (t', s) satisfies the condition (1), then the assertion is obvious. Therefore we assume that (t', s) satisfies the condition (2). If $s_0 = 0$, then we have $s_{-1} = -1 = t_{-1}$ and

$$j_{2s_0} = j_{2s'_0} \geq i_{2t'_0} \geq i_{2t_0}.$$

Then it is easy to check the condition (2) of this lemma. Hence we may assume that $s_0 > 0$.

Claim 3. *We have the following.*

(1) *There exists $\ell \geq 0$ such that*

$$\begin{cases} t'_{r-\ell} = t_r & r \leq -1 \\ s'_{r-\ell} = s_r & r \leq -1 \end{cases}$$

(2) *If $t_0 \neq t'_{-\ell}$, then we have*

$$j_{2s_0} \geq j_{2s_{-1}+2} \geq i_{2t_0} > i_{2t_0-1} \geq j_{2s_{-1}+1}.$$

In particular, (t, s) satisfies the condition (2).

Proof. Note that $s'_{-1} \geq s_{-1}$. First we assume that $s'_{-1} = s_{-1}$. If $s'_{-1} = s_{-1} = -1$, then $t'_{-1} = t_{-1} = -1$. If $s'_{-1} = s_{-1} \geq 0$, then we have that

$$i_{2t'_0} > \cdots > i_{2t_0} \geq i_{2t_{-1}+2} > j_{2s'_{-1}+1} = j_{2s_{-1}+1} > i_{2t_{-1}}.$$

Therefore we obtain $t'_{-1} = t_{-1}$. Hence the assertion is obvious. Next we assume that $s'_{-1} > s_{-1}$. Let ℓ be a positive integer such that $j_{2s'_{-\ell}+1} \geq j_{2s_{-1}+3} \geq j_{2s'_{-\ell-1}+3}$. Thus we have that

$$i_{2t'_{-\ell}+2} > j_{2s'_{-\ell}+1} \geq j_{2s_{-1}+3} \geq j_{2s'_{-\ell-1}+3} > i_{2t'_{-\ell}}.$$

Note that $j_{2s_{-1}+3} > i_{2t_0} > j_{2s_{-1}+1}$. If $j_{2s_{-1}+3} > j_{2s'_{-\ell-1}+3}$, then $j_{2s_{-1}+1} \geq j_{2s'_{-\ell-1}+3}$ and so we have

$$i_{2t'_{-\ell}+2} > j_{2s'_{-\ell}+1} \geq j_{2s_{-1}+3} > i_{2t_0} > j_{2s_{-1}+1} \geq j_{2s'_{-\ell-1}+3} > i_{2t'_{-\ell}}.$$

This gives that

$$t'_{-\ell} < t_0 < t'_{-\ell} + 1.$$

This is a contradiction. Hence we conclude that

$$s_{-1} = s'_{-\ell-1}.$$

If $i_{2t_0} > i_{2t'_{-\ell}}$, then we get that

$$i_{2t'_{-\ell}+2} > j_{2s'_{-\ell}+1} \geq j_{2s_{-1}+3} > i_{2t_0} > i_{2t'_{-\ell}}.$$

This is a contradiction. Therefore, we obtain that

$$i_{2t_0} \leq i_{2t'_{-\ell}}.$$

Then we have that

$$i_{2t'_{-\ell}} \geq i_{2t_0} \geq i_{2t_{-1}+2} > j_{2s'_{-\ell-1}+1} = j_{2s_{-1}+1} > i_{2t_{-1}}.$$

This implies $t'_{-\ell-1} = t_{-1}$. (Remark: If $s_{-1} = s'_{-\ell-1} = -1$, then $t'_{-\ell-1} = t_{-1} = -1$.) In particular, we conclude that

$$\begin{cases} t'_{r-\ell} = t_r & r \leq -1 \\ s'_{r-\ell} = s_r & r \leq -1 \end{cases}$$

Suppose that $t_0 \neq t'_{-\ell}$. In this case, we have $i_{2t'_{-\ell}} > i_{2t_0}$. Since (t', s) satisfies the condition (2), we conclude that

$$j_{2s_0} \geq j_{2s_{-1}+2} = j_{2s'_{-\ell-1}+2} \geq i_{2t'_{-\ell}} > i_{2t_0} > j_{2s'_{-\ell-1}+1} = j_{2s_{-1}+1}.$$

By applying Lemma 4.3 (3) for $f_{i_{2t_0}}^{j_{2s'_{-\ell-1}+1}}$, we obtain that

$$i_{2t_0-1} \geq j_{2s'_{-\ell-1}+1} = j_{2s_{-1}+1}.$$

■

Therefore by Claim 3, the assertion also holds for the case that $2t_1 - 2 > 2t$. □

For $\mathbf{i}, \mathbf{j} \in \Xi$ and $f : P_{i_{2t}} \rightarrow P_{j_{2s+1}}$, we set $\varphi(\mathbf{i}, \mathbf{j}, f) = \varphi(f) := (\varphi_{i_{2p}}^{j_{2q+1}} : P_{i_{2p}} \rightarrow P_{j_{2q+1}}) \in \text{Hom}_{\Lambda}(X_{\mathbf{i}}^{-1}, X_{\mathbf{j}}^0)$ where

$$\varphi_{i_{2p}}^{j_{2q+1}} = \begin{cases} f & (p, q) = (t, s) \\ 0 & \text{otherwise.} \end{cases}$$

We regard $\varphi(t, s, f)$ as a morphism in $\text{Hom}_{\mathbf{K}^b(\text{proj } \Lambda)}(X_{\mathbf{i}}, X_{\mathbf{j}}[1])$.

Lemma 4.8. *Let $\mathbf{i}, \mathbf{j} \in \Xi$. Let (t, s) be a pair such that $0 < i_{2t} < j_{2s+1} < n + 1$. If either (1) or (2) of Lemma 4.7 holds, then $\varphi(f) = 0$ for any $f \in \text{Hom}_{\Lambda}(P_{i_{2t}}, P_{j_{2s+1}})$.*

Proof. We first consider the case that the assertion (1) of Lemma 4.7 holds. We note that $i_{2t+1} = i_{2t_0+1} \leq j_{2s_0+1} = j_{2s+1} \leq n$. In particular, we have $t < m_{\mathbf{i}}$ and $s < m_{\mathbf{j}}$.

Claim 4. *We have the following.*

- (1) $t_1 \leq m_{\mathbf{i}}$ and $s_1 \leq m_{\mathbf{j}}$.
- (2) If $t_1 \leq m_{\mathbf{i}}$ and $s_1 < m_{\mathbf{j}}$, then there exists $h_{i_{2p+1}}^{j_{2s+1}} : P_{i_{2p+1}} \rightarrow P_{j_{2s+1}}$ and $h_{i_{2t_1}}^{j_{2q+2}} : P_{i_{2t_1}} \rightarrow P_{j_{2q+1}}$ for any $p \in \{t, \dots, t_1 - 1\}$ and $q \in \{s, \dots, s_1 - 1\}$ such that

$$\varphi(f) - h \circ d_{\mathbf{i}} - d_{\mathbf{j}} \circ h' = \varphi(g)$$

for some $g : P_{2t_1} \rightarrow P_{2s_1+1}$, where $h \in \text{Hom}_{\Lambda}(X_{\mathbf{i}}^0, X_{\mathbf{j}}^0)$ and $h' \in \text{Hom}_{\Lambda}(X_{\mathbf{i}}^{-1}, X_{\mathbf{j}}^{-1})$ are morphisms given by $\{h_{i_{2p+1}}^{j_{2s+1}} \mid t \leq p \leq t_1 - 1\}$ and $\{h_{i_{2t_1}}^{j_{2q+2}} \mid s \leq q \leq s_1 - 1\}$ respectively.

- (3) If $i_{2t_1} = n + 1$, then $\varphi(f) = 0$ in $\mathbf{K}^b(\text{proj } \Lambda)$.
- (4) If $s_1 = m_{\mathbf{j}}$, then $\varphi(f) = 0$ in $\mathbf{K}^b(\text{proj } \Lambda)$.
- (5) If $t_1 = m_{\mathbf{i}}$, then $s_1 = m_{\mathbf{j}}$.

Proof. We show (1). By the condition (1)-(b) of Lemma 4.7 (note that $t_1 \leq m_{\mathbf{i}} + 1$ by definition of t_1), we get that

$$i_{2t_1-1} \leq j_{2s+1} \leq n.$$

This implies that $t_1 \leq m_{\mathbf{i}}$. Then $s_1 \leq m_{\mathbf{j}}$ follows from the definition of s_1 . We prove (2) and (3). By Lemma 4.3, we can write $f = l \circ f_{i_{2t}}^{j_{2s+1}}$ for some $l \in \text{End}_{\Lambda}(P_{j_{2s+1}})$. By the condition (1)-(a) of Lemma 4.7, we conclude that $w_{i_{2t}}^{j_{2s+1}}$ factors through i_{2t_0+1} . In particular, we obtain that

$$\varphi(f) - h_1 \circ d_{\mathbf{i}} = \varphi(f_1),$$

for some $f_1 : P_{i_{2t+2}} \rightarrow P_{j_{2s+1}}$, where h_1 is a morphism in $\text{Hom}_{\mathbf{K}^b(\text{proj } \Lambda)}(X_{\mathbf{i}}^0, X_{\mathbf{j}}^0)$ given by $h_{i_{2t+1}}^{j_{2s+1}} := l \circ f_{i_{2t+1}}^{j_{2s+1}}$ and $f_1 : P_{i_{2t+2}} \rightarrow P_{j_{2s+1}}$. Inductively, one can construct $h_{i_{2p+1}}^{j_{2s+1}} : P_{i_{2p+1}} \rightarrow P_{j_{2s+1}}$ for any $p \in \{t, t+1, \dots, t_1 - 2\}$ and $f' : P_{i_{2t_1-2}} \rightarrow P_{j_{2s+1}}$ such that

$$\varphi(t, s, f) - h' \circ d_{\mathbf{i}} = \varphi(f'),$$

where h' is a morphism in $\text{Hom}_{\mathbf{K}^b(\text{proj } \Lambda)}(X_{\mathbf{i}}^0, X_{\mathbf{j}}^0)$ given by $\{h_{i_{2p+1}}^{j_{2s+1}} \mid t \leq p \leq t_1 - 2\}$. By using the condition (1)-(b) of Lemma 4.7, there are $h_{i_{2t_1-1}}^{j_{2s+1}} : P_{i_{2t_1-1}} \rightarrow P_{j_{2s+1}}$ and $f^{(1)} : P_{i_{2t_1}} \rightarrow P_{j_{2s+1}}$ such that

$$\varphi(f) - h \circ d_{\mathbf{i}} = \varphi(f^{(1)}),$$

where h is a morphism in $\text{Hom}_{\mathbf{K}^b(\text{proj } \Lambda)}(X_{\mathbf{i}}^0, X_{\mathbf{j}}^0)$ given by $\{h_{i_{2p+1}}^{j_{2s+1}} \mid t \leq p \leq t_1 - 1\}$. Thus we have $\varphi(f) - h \circ d_{\mathbf{i}} = 0$ in the case that $i_{2t_1} = n + 1$ and get the assertion (3). Assume that

$i_{2t_1} < n + 1$. Lemma 4.3 implies that there exists $l' \in \text{End}_\Lambda(P_{i_{2t_1}})$ such that $f^{(1)} = f_{i_{2t_1}}^{j_{2s+1}} \circ l'$. Now by the condition (1)-(c) of Lemma 4.7, we obtain that $w_{i_{2t_1}}^{j_{2s+1}}$ factors through j_{2s+2} . Therefore, we conclude that

$$\varphi(f^{(1)}) - d_j \circ h'_1 = \varphi(f_1^{(1)}),$$

for some $f_1^{(1)} : P_{i_{2t+2}} \rightarrow P_{j_{2s+1}}$, where h'_1 is a morphism in $\text{Hom}_{\mathbf{K}^b(\text{proj } \Lambda)}(X_i^{-1}, X_j^{-1})$ given by $h_{i_{2t_1}}^{j_{2s+2}} := f_{i_{2t_1}}^{j_{2s+2}} \circ l'$. Inductively, (and by using the condition (1)-(d) of Lemma 4.7), we can construct $h_{i_{2t_1}}^{j_{2q+2}} : P_{i_{2t_1}} \rightarrow P_{j_{2q+2}}$ for any $q \in \{s, s+1, \dots, s_1-1\}$ and $g : P_{i_{2t_1}} \rightarrow P_{j_{2s+1}}$ such that

$$\varphi(f^{(1)}) - d_j \circ h = \varphi(g),$$

where h' is a morphism in $\text{Hom}_{\mathbf{K}^b(\text{proj } \Lambda)}(X_i^{-1}, X_j^{-1})$ given by $\{h_{i_{2t_1}}^{j_{2q+2}} \mid s \leq q \leq s_1-1\}$.

We prove (4). By the assertion (3), we may assume that $i_{2t_1} \leq n$. Then $j_{2s_1} \leq i_{2t_1} \leq n$ follows from the condition (1)-(d) of Lemma 4.7. Therefore, for a morphisms $h \in \text{Hom}_\Lambda(X_i^0, X_j^0)$ and $h' \in \text{Hom}_\Lambda(X_i^{-1}, X_j^{-1})$ constructed in the proof of (2), we have that

$$\varphi(f) - h \circ d_i - d_j \circ h' = 0.$$

Finally we show the assertion (5). The condition (1)-(a) of Lemma 4.7 implies that

$$n + 2 = i_{2t_1+1} \leq j_{2s_1+1}.$$

Hence $s_1 = m_j$. ■

Therefore if $i_{2t} < j_{2s+1}$ and the assertion (1) of Lemma 4.7 hold, then $\varphi(f) = 0$ follows from Claim 4.

Next we assume that the condition (2) of Lemma 4.7 holds. We note that $j_{2s} = j_{2s_0} \geq i_{2t_0} = i_{2t} \geq 1$.

Claim 5. *We have the following.*

- (1) $s_{-1}, t_{-1} \geq -1$.
- (2) $j_{2s_{-1}+2} > 0 (\Leftrightarrow P_{j_{2s_{-1}+2}} \neq 0)$.
- (3) If $s_{-1} \geq 0$ and $t_{-1} \geq 0$, then there exists $h_{i_{2t}}^{j_{2q}} : P_{i_{2t}} \rightarrow P_{j_{2q}}$ and $h_{i_{2p-1}}^{j_{2s_{-1}+1}} : P_{i_{2p-1}} \rightarrow P_{j_{2s_{-1}+1}}$ for any $q \in \{s, \dots, s_{-1}+1\}$ and $p \in \{t, \dots, t_{-1}+1\}$ such that

$$\varphi(f) - d_j \circ h' - h \circ d_i = \varphi(g)$$

for some $g : P_{i_{2t-1}} \rightarrow P_{j_{2s_{-1}+1}}$, where $h' \in \text{Hom}_\Lambda(X_i^{-1}, X_j^{-1})$ and $h \in \text{Hom}_\Lambda(X_i^0, X_j^0)$ are morphisms given by $\{h_{i_{2t}}^{j_{2q}} \mid s \geq q \geq s_{-1}+1\}$ and $\{h_{i_{2p-1}}^{j_{2s_{-1}+1}} \mid t \geq p \geq t_{-1}+1\}$ respectively.

- (4) If $s_{-1} = -1$, then $\varphi(f) = 0$ in $\mathbf{K}^b(\text{proj } \Lambda)$.
- (5) If $t_{-1} = -1$, then $s_{-1} = -1$.

Proof. The assertion (1) follows from the definitions of s_{-1} and t_{-1} . The condition (2)-(b') of Lemma 4.7 gives that $j_{2s_{-1}+2} \geq i_{2t_0} > 0$. Hence we obtain the assertion (2). The assertion (5) follows from (2)-(d') of Lemma 4.7. In fact, if $t_{-1} = -1$, then $-1 = i_{2t_{-1}+1} \geq j_{2s_{-1}+1}$. One can apply similar argument used in the proof of Claim 4 (2),(3) and get the assertion (3) and (4). ■

Then $\varphi(f) = 0$ directly follows from Claim 5. □

By considering labeling-change $i \leftrightarrow n+1-i$ on $Q_0 \sqcup \{0, n+1\} = \{0, 1, \dots, n, n+1\}$, we also obtain the following.

Lemma 4.9. *Let $\mathbf{i}, \mathbf{j} \in \Xi$. For a pair (t, s) such that $n+1 > i_{2t} > j_{2s-1} > 0$, we define two sequences $\mathbf{t}^+(t, s) := (t = t_0 \geq t_1 \geq t_2 \geq \dots)$ and $\mathbf{s}^+(t, s) := (s = s_0 \geq s_1 \geq s_2 \dots)$ as follows:*

$$\begin{aligned} \text{(i)} \quad t_r &:= \begin{cases} \min\{t \geq -1 \mid i_{2t+2} > j_{2s_{r-1}-1}\} & \text{if } s_{r-1} \geq 1 \\ -1 & \text{if } s_{r-1} \leq 0, t_{r-1} \geq 0 \\ -2 & \text{if } t_{r-1} \leq -1 \end{cases} \\ \text{(ii)} \quad s_r &:= \begin{cases} \min\{s \geq 0 \mid j_{2s+1} > i_{2t_r}\} & \text{if } t_r \geq 0 \\ 0 & \text{if } t_r = -1, s_{r-1} \geq 1 \\ -1 & \text{if } s_{r-1} \leq 0 \end{cases} \end{aligned}$$

Also we define two sequences $\mathbf{t}^-(t, s) := (t = t_0 \leq t_{-1} \leq t_{-2} \leq \dots)$ and $\mathbf{s}^-(t, s) := (s = s_0 \leq s_{-1} \leq s_{-2} \leq \dots)$ as follows:

$$\begin{aligned} \text{(iii)} \quad s_r &:= \begin{cases} \max\{s \leq m_{\mathbf{j}} + 1 \mid j_{2s-3} < i_{2t_{r+1}}\} & \text{if } t_{r+1} \leq m_{\mathbf{i}} \\ m_{\mathbf{j}} + 1 & \text{if } t_{r+1} = m_{\mathbf{i}} + 1, s_{r+1} \leq m_{\mathbf{j}} \\ m_{\mathbf{j}} + 2 & \text{if } s_{r-1} \geq m_{\mathbf{j}} + 1 \end{cases} \\ \text{(iv)} \quad t_r &:= \begin{cases} \max\{t \leq m_{\mathbf{i}+1} \mid i_{2t-2} < j_{2s_{r-1}}\} & \text{if } s_r \leq m_{\mathbf{j}} \\ m_{\mathbf{i}} + 1 & \text{if } s_r \geq m_{\mathbf{j}} + 1, t_{r+1} \leq m_{\mathbf{i}} \\ m_{\mathbf{i}} + 2 & \text{if } t_{r+1} \geq m_{\mathbf{i}} + 1 \end{cases} \end{aligned}$$

If $\text{Hom}_{\mathbf{K}^b(\text{proj } \Lambda)}(X_{\mathbf{i}}, X_{\mathbf{j}}[1]) = 0$, then one of the following holds.

- (1) We have (a), (b), (c) and (d).
 - (a) $i_{2t_r+2} > i_{2t_r+1} \geq j_{2s_{r-1}-1}$ for any $r \geq 1$ such that $t_r \geq -1$.
 - (b) $j_{2s_{r-1}-1} > j_{2s_{r-1}-2} \geq i_{2t_r}$ for any $r \geq 1$ such that $s_r \geq 0$.
 - (c) $j_{2s_r+1} > j_{2s_r} \geq i_{2t_r}$ for any $r \geq 1$ such that $s_r \geq 0$.
 - (d) $i_{2t_r} > i_{2t_r-1} \geq j_{2s_{r-1}}$ for any $r \geq 0$ such that $t_r \geq 0$.
 Where we put $i_{-1} = j_{-1} = -1$ and $i_{-2} = j_{-2} = -2$.
- (2) We have (a'), (b'), (c') and (d').
 - (a') $j_{2s_r-3} < j_{2s_r-2} \leq i_{2t_{r+1}}$ for any $r \leq -1$ such that $s_r \leq m_{\mathbf{j}} + 1$.
 - (b') $i_{2t_{r+1}} < i_{2t_{r+1}+1} \leq j_{2s_{r-1}}$ for any $r \leq -1$ such that $t_r \leq m_{\mathbf{i}} + 1$.
 - (c') $i_{2t_r-2} < i_{2t_r-1} \leq j_{2s_{r-1}}$ for any $r \leq -1$ such that $t_r \leq m_{\mathbf{i}} + 1$.
 - (d') $j_{2s_{r-1}} < j_{2s_r} \leq i_{2t_r}$ for any $r \leq -1$ such that $s_r \leq m_{\mathbf{j}}$.
 Where we put $i_{2m_{\mathbf{i}}+1} = j_{2m_{\mathbf{j}}+1} = n+2$ and $i_{2m_{\mathbf{i}}+2} = j_{2m_{\mathbf{j}}+2} = n+3$.

Lemma 4.10. *Let $\mathbf{i}, \mathbf{j} \in \Xi$. Let (t, s) be a pair such that $n+1 > i_{2t} > j_{2s-1} > 0$. If either (1) or (2) of Lemma 4.9 holds, then $\varphi(f) = 0$ for any $f \in \text{Hom}_{\Lambda}(P_{i_{2t}}, P_{j_{2s-1}})$.*

By Lemma 4.7, Lemma 4.8, Lemma 4.9 and Lemma 4.10, we obtain a combinatorial description of $\text{Hom}_{\mathbf{K}^b(\text{proj } \Lambda)}(X_{\mathbf{i}}, X_{\mathbf{j}}[1]) = 0$.

Proposition 4.11. *Let $\mathbf{i}, \mathbf{j} \in \Xi$. Then $\text{Hom}_{\mathbf{K}^b(\text{proj } \Lambda)}(X_{\mathbf{i}}, X_{\mathbf{j}}[1]) = 0$ if and only if the following two conditions hold.*

- (a) For any pair (t, s) with $0 < i_{2t} < j_{2s+1} < n+1$, either (1) or (2) of Lemma 4.7 holds.
- (b) For any pair (t, s) with $0 > i_{2t} > j_{2s-1} > n+1$, either (1) or (2) of Lemma 4.9 holds.

Let $\mathbb{P}_{\Lambda} := \text{st-tilt } \Lambda \cap \text{add } \bigoplus_{\mathbf{i} \in \Xi} X_{\mathbf{i}}$.

Lemma 4.12. *Let Λ be the preprojective algebra of type A_n . Then $\mathbb{P}_\Lambda = \text{s}\tau\text{-tilt } \Lambda$.*

Note that the number of isomorphism classes of indecomposable 2-term presilting object of $\text{K}^b(\text{proj } \Lambda)$ is equal to $\# \tau\text{-rigid } \Lambda + n$. Then Lemma 4.5 implies that

$$\# \tau\text{-rigid } \Lambda + n \leq \#\{\mathbf{i} \in \Xi \mid m_{\mathbf{i}} > 1\} + n = \#\Xi.$$

By Lemma 4.6, Proposition 4.11, we have that $X_{\mathbf{i}}$ is indecomposable 2-term silting object of $\text{K}^b(\text{proj } \Lambda)$ for any $\mathbf{i} \in \Xi$. Hence X is an indecomposable 2-term presilting object of $\text{K}^b(\text{proj } \Lambda)$ if and only if X is isomorphic to $X_{\mathbf{i}}$ for some $\mathbf{i} \in \Xi$.

Proposition 4.13. *Assume that Λ satisfies the Condition 3.1. Then we have*

$$\text{s}\tau\text{-tilt } \Lambda \simeq (\mathfrak{S}_{n+1}, \leq).$$

Proof. By Lemma 4.6, Proposition 4.11 and Lemma 4.12, we have that \mathbb{P}_Λ is a full subposet of $\text{s}\tau\text{-tilt } \Lambda$ and

$$\mathbb{P}_\Lambda \simeq (\mathfrak{S}_{n+1}, \leq).$$

This gives the assertion. In fact, \mathbb{P}_Λ is n -regular and so finite connected component of $\text{s}\tau\text{-tilt } \Lambda$. Hence we have $\mathbb{P}_\Lambda = \text{s}\tau\text{-tilt } \Lambda$. \square

4.3. ‘only if’ part. In this subsection, we prove that $\text{s}\tau\text{-tilt } \Lambda \simeq (\mathfrak{S}_{n+1}, \leq)$ induces the Condition 3.1. For a proof, we use an induction on n . In subsection 4.1, we treated the case $n = 2$ and in subsection 4.2, we showed that $\text{s}\tau\text{-tilt } \Lambda \simeq (\mathfrak{S}_{n+1}, \leq)$ if Λ satisfies the Condition 3.1. Hence we may assume that Theorem 3.3 holds for any Λ such that $\#Q_0 < n$ and consider the case that $\#Q_0 = n$.

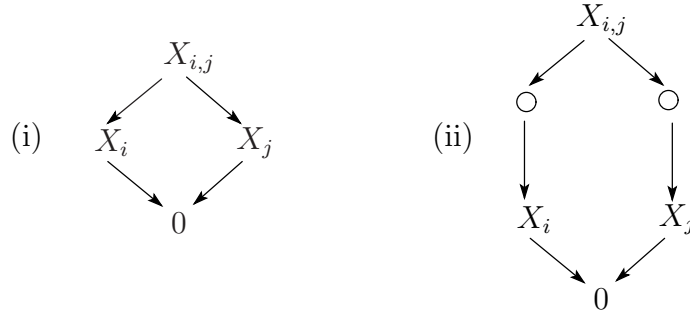
Let $\Lambda = kQ/I$ be an algebra such that

$$\text{s}\tau\text{-tilt } \Lambda \simeq (\mathfrak{S}_{n+1}, \leq).$$

Let $\rho : (\mathfrak{S}_{n+1}, \leq) \xrightarrow{\sim} \text{s}\tau\text{-tilt } \Lambda$ and denote by $T_w := \rho(w)$. For any $a \in Q_0$, we set $X_a := e_a \Lambda / e_a \Lambda (1 - e_a) \Lambda$. Then $\text{dp}(0) = \{X_a \mid a \in Q_0\} = \{T_{s_i} \mid 1 \leq i \leq n\}$. Therefore we may assume that $Q_0 = \{1, \dots, n\}$ and $T_{s_i} = X_i$.

Lemma 4.14. *For $i \neq j \in Q_0$, we set $X_{i,j} := X_i \vee X_j$.*

(1) $[0, X_{i,j}]$ has one of the following form.



(2) In the case of (i), there is no edge between i and j in Q .

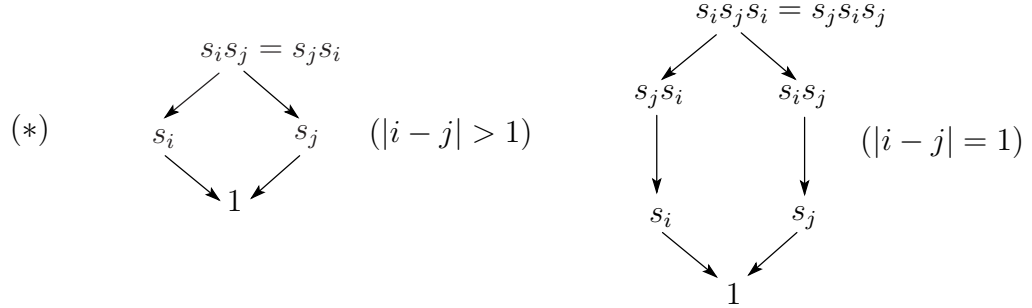
(3) In the case of (ii), then we have

$$i \rightleftarrows j$$

(4) Q° is a double quiver of type A_n , i.e.

$$Q^\circ = 1 \rightleftarrows 2 \rightleftarrows 3 \rightleftarrows \cdots \rightleftarrows n$$

Proof. Note that $[1, s_i \vee s_j]$ has following form.



Accordingly, we have (1).

We prove remaining assertions. If $[0, X_{i,j}]$ has form (i), then X_i and X_j are projective $\Lambda/(1 - e_i - e_j)$ -modules. Therefore, we obtain the assertion (2). Let M be a maximum element of $\mathbf{s\tau\text{-tilt}}_{(1-e_i-e_j)\Lambda^-} \Lambda$. Then $X_i, X_j \leq M$ implies that $X_{i,j} \leq M$. In particular, $[0, X_{i,j}]$ is a full subposet of $\mathbf{s\tau\text{-tilt}}_{(1-e_i-e_j)\Lambda^-} \Lambda \simeq \mathbf{s\tau\text{-tilt}} \Lambda / (1 - e_i - e_j)$. Since $[0, X_{i,j}]$ is 2-regular and so finite connected component of $\mathbf{s\tau\text{-tilt}}_{(1-e_i-e_j)\Lambda^-} \Lambda$. This implies that

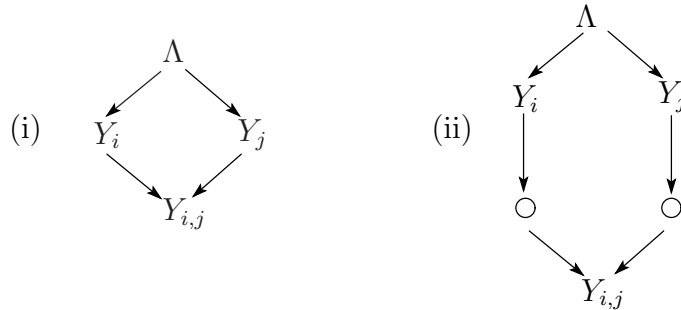
$$[0, X_{i,j}] = \mathbf{s\tau\text{-tilt}}_{(1-e_i-e_j)\Lambda^-} \Lambda \simeq \mathbf{s\tau\text{-tilt}} \Lambda / (1 - e_i - e_j).$$

Then the assertion (3) follows from Proposition 4.2. The assertion (4) is a direct consequence of (1), (2), (3) and (*). \square

Lemma 4.15. *Let $Y_i \in \mathbf{ds}(\Lambda)$ such that $P_i \notin \mathbf{add} Y_i$. We define $\sigma \in \mathfrak{S}_n$ as follows:*

$$T_{s_i w_0} = Y_{\sigma(i)}.$$

(1) *Let $i \neq j \in Q_0$ and $Y_{i,j} := Y_i \wedge Y_j$. Then $[Y_{i,j}, \Lambda]$ has one of the following form.*



(2) *In the case of (i), there is no edge between i and j in Q .*

(3) *In the case of (ii), then we have*

$$i \rightleftarrows j$$

(4) *σ induces quiver automorphism*

$$Q^\circ \xrightarrow{\sim} Q^\circ.$$

Proof. Similar to the proof of Lemma 4.14 (1), one can easily check (1). Note that $Y_i, Y_j \in \mathbf{s\tau\text{-}tilt}_{\Lambda/(e_i\Lambda \oplus e_j\Lambda)} \Lambda \simeq \mathbf{s\tau\text{-}tilt} \Lambda/(1 - e_i - e_j)$. By using same argument used in the proof of Lemma 4.14, one has $[Y_i \wedge Y_j, \Lambda] = \mathbf{s\tau\text{-}tilt}_{\Lambda/(e_i\Lambda \oplus e_j\Lambda)} \Lambda$. Then the assertions (2) and (3) follow from Proposition 4.2. We prove (4). i and j shear an edge in Q if and only if $|i - j| = 1$. By definition of σ , we have that $|i - j| = 1$ if and only if $Y_{\sigma(i)}$ and $Y_{\sigma(j)}$ satisfies (ii). Hence the assertion follows from (1), (2) and (3). \square

Lemma 4.16. *Let $i < j$ and $e = e_i + e_{i+1} + \cdots + e_j$.*

- (1) $\mathbf{s\tau\text{-}tilt}(\Lambda/(1 - e)) \simeq (\mathfrak{S}_{j-i+2}, \leq)$.
- (2) *If $j - i < n - 1$, then we have a path*

$$e_i\Lambda/e_i\Lambda(1-e_i)\Lambda \leftarrow e_i\Lambda/e_i\Lambda(1-e_i)\Lambda \oplus e_i\Lambda/e_i\Lambda(1-e_i-e_{i+1})\Lambda \leftarrow \cdots \leftarrow \bigoplus_{k=i}^j e_i\Lambda/e_i\Lambda(1-e_i-\cdots-e_k)\Lambda$$

in $\mathbf{s\tau\text{-}tilt}_{(1-e)\Lambda^-} \Lambda$.

Proof. We prove (1). Since $s_i \vee s_{i+1} \vee \cdots \vee s_j$ be the longest element in $\langle s_i, \dots, s_j \rangle \simeq \mathfrak{S}_{j-i+2}$, we have that

$$[0, T_{s_i} \vee \cdots \vee T_{s_j}] = \rho([1, s_i \vee \cdots \vee s_j]) \simeq (\mathfrak{S}_{j-i+2}, \leq).$$

Note that $T_{s_i}, \dots, T_{s_j} \in \mathbf{s\tau\text{-}tilt}_{(1-e)\Lambda^-} \Lambda$. Therefore for a maximum element M of $\mathbf{s\tau\text{-}tilt}_{(1-e)\Lambda^-} \Lambda$, we obtain that

$$T_{s_i} \vee \cdots \vee T_{s_j} \leq M.$$

In particular,

$$[0, T_{s_i} \vee \cdots \vee T_{s_j}] \subset \mathbf{s\tau\text{-}tilt}_{(1-e)\Lambda^-} \Lambda.$$

Since $\mathbf{s\tau\text{-}tilt}_{(1-e)\Lambda^-} \Lambda \simeq \mathbf{s\tau\text{-}tilt}(\Lambda/(1 - e))$ is $(j - i + 1)$ -regular poset, we conclude that

$$\mathbf{s\tau\text{-}tilt}(\Lambda/(1 - e)) \simeq [0, T_{s_i} \vee \cdots \vee T_{s_j}] \simeq (\mathfrak{S}_{j-i+2}, \leq).$$

Next we show (2). By (1) and the hypothesis of induction, $\Lambda_e := \Lambda/(1 - e)$ satisfies (a), (b) and (c) of Condition 3.1. Hence one can check that

$$\bigoplus_{k=i}^{\ell-1} e_i\Lambda_e/e_i\Lambda_e(1 - e_i - \cdots - e_k)\Lambda_e \leftarrow \bigoplus_{k=i}^{\ell} e_i\Lambda_e/e_i\Lambda_e(1 - e_i - \cdots - e_k)\Lambda_e.$$

In fact, the 2-term presilting object in $\mathbf{K}^b(\text{proj } \Lambda_e)$ corresponding to $e_i\Lambda_e/e_i\Lambda_e(1 - e_{i,k})\Lambda$ is $X_{\mathbf{i}_{i,k}}$, where $e_{i,k} := e_i + \cdots + e_k$ and $\mathbf{i}_{i,k} := \{i - 1 < i < k + 1\}$. Then $X_{\mathbf{i}_{i,k}} \oplus X_{\mathbf{i}_{i,k'}}$ is a presilting object of $\mathbf{s\tau\text{-}tilt} \Lambda_e = \mathbf{s\tau\text{-}tilt}_{(1-e)\Lambda^-} \Lambda$ by Proposition 4.11 (or direct calculation). Then the assertion follows from

$$e_i\Lambda_e/e_i\Lambda_e(1 - e_i - \cdots - e_k)\Lambda_e \simeq e_i\Lambda/e_i\Lambda(1 - e_i - \cdots - e_k)\Lambda.$$

\square

Lemma 4.17. *We have the following.*

- (1) *If $i < j$. Then $T_{s_j s_{j-1} \cdots s_i}$ is a unique element of*

$$\text{dp}(T_{s_{j-1} \cdots s_i}) \cap [T_{s_{j-2} \cdots s_i}, T_{s_{j-1} \cdots s_i} \vee T_{s_j}].$$

Moreover, if $(i, j) \neq (1, n)$, then

$$T_{s_j s_{j-1} \cdots s_i} = \bigoplus_{k=i}^j e_i \Lambda / e_i \Lambda (1 - e_i - \cdots - e_k) \Lambda.$$

(2) If $i > j$. Then $T_{s_j s_{j+1} \cdots s_i}$ is a unique element of

$$\text{dp}(T_{s_{j+1} \cdots s_i}) \cap [T_{s_{j+2} \cdots s_i}, T_{s_{j+1} \cdots s_i} \vee T_{s_j}].$$

Moreover, if $(i, j) \neq (n, 1)$, then

$$T_{s_j s_{j+1} \cdots s_i} = \bigoplus_{k=i}^j e_i \Lambda / e_i \Lambda (1 - e_i - \cdots - e_k) \Lambda.$$

Proof. We consider the case that $i < j$. We claim that

$$T_{s_j s_{j-1} \cdots s_i} \in \text{dp}(T_{s_{j-1} \cdots s_i}) \cap [T_{s_{j-2} \cdots s_i}, T_{s_{j-1} \cdots s_i} \vee T_{s_j}].$$

It is obvious that $T_{s_j s_{j-1} \cdots s_i} \in \text{dp}(T_{s_{j-1} \cdots s_i})$. Put $w = s_{j-2} \cdots s_i$. Then we obtain that

$$\begin{array}{ccc} & s_j s_{j-1} s_j w & \\ \swarrow & & \searrow \\ s_{j-1} s_j w & & s_j s_{j-1} w \\ \downarrow & & \downarrow \\ ws_j = s_j w & & s_{j-1} w \\ \searrow & & \swarrow \\ & w & \end{array}$$

Note that $ws_j \geq s_j$ and $w \not\geq s_j$. Hence we conclude that $ws_j = s_j \vee w$ and

$$s_{j-1} w \vee s_j = s_{j-1} w \vee w \vee s_j = s_j s_{j-1} s_j w = s_{j-1} s_j s_{j-1} w.$$

In particular, we conclude that

$$T_{s_j s_{j-1} w} \in [T_w, T_{s_{j-1} w} \vee T_{s_j w}] = [T_w, T_{s_{j-1} w} \vee T_{s_j}].$$

Then the uniqueness follows from the fact that $\text{ds}(s_j s_{j-1} s_j w) = \{s_{j-1} s_j w, s_j s_{j-1} w\}$.

We note that $T_{s_i} = X_i = e_i \Lambda / e_i \Lambda (1 - e_i) \Lambda$. Now we assume that

$$T_{s_{j'} s_{j'-1} \cdots s_i} = \bigoplus_{k=i}^{j'} e_i \Lambda / e_i \Lambda (1 - e_i - \cdots - e_{j'}) \Lambda$$

holds for any $j' \in \{i, \dots, j-1\}$. Then $T_{s_j w} = T_w \vee T_{s_j} = T_w \oplus X_j$ and $T_{s_{j-1} w} = T_w \oplus e_i \Lambda / e_i \Lambda (1 - e_i - \cdots - e_{j-1}) \Lambda$. Therefore $T_{s_j w}, T_{s_{j-1} w} \in \mathbf{s\tau\text{-tilt}}_{T_w \oplus (1-e) \Lambda^-} \Lambda$, where $e := e_i + \cdots + e_j$. This shows that $T_{s_j w} \vee T_{s_{j-1} w} \leq M$, where M is a maximum element of $\mathbf{s\tau\text{-tilt}}_{T_w \oplus (1-e) \Lambda^-} \Lambda$. In particular, we have that

$$\mathbf{s\tau\text{-tilt}}_{T_w \oplus (1-e) \Lambda^-} \Lambda \supset [T_w, T_{s_{j-1} w} \vee T_{s_j w}] = [T_w, T_{s_{j-1} w} \vee T_{s_j}].$$

By Jasso's theorem, we have that $\mathbf{s\tau\text{-tilt}}_{T_w \oplus (1-e) \Lambda^-} \Lambda$ is a two-regular poset. Hence we obtain that

$$\mathbf{s\tau\text{-tilt}}_{T_w \oplus (1-e) \Lambda^-} \Lambda = [T_w, T_{s_{j-1} w} \vee T_{s_j w}] = [T_w, T_{s_{j-1} w} \vee T_{s_j}].$$

Note that Lemma 4.16 implies

$$T_{s_{j-1}w} \oplus e_i \Lambda / e_i \Lambda (1 - e) \Lambda \in \mathbf{s\tau\text{-}tilt}_{T_w \oplus (1-e)\Lambda} \Lambda.$$

Hence, we get that

$$\bigoplus_{k=i}^j e_i \Lambda / e_i \Lambda (1 - e_i - \cdots - e_k) \Lambda = T_{s_{j-1}w} \oplus e_i \Lambda / e_i \Lambda (1 - e) \Lambda \in \mathbf{dp}(T_{s_{j-1}w}) \cap [T_w, T_{s_{j-1}w} \vee T_{s_j}].$$

In particular, the following hold.

$$T_{s_j s_{j-1} w} = \bigoplus_{k=i}^j e_i \Lambda / e_i \Lambda (1 - e_i - \cdots - e_k) \Lambda.$$

Accordingly, we obtain (1). Similar argument gives the assertion (2). \square

Lemma 4.18. *Let $w = s_n s_{n-1} \cdots s_i$.*

(1) *For any $\sigma \in \langle s_1, \dots, s_{n-1} \rangle$, we have*

$$\sigma w \geq w.$$

(2) *Let $\sigma, \sigma' \in \langle s_1, \dots, s_{n-1} \rangle$. Then*

$$\sigma w \leq \sigma' w \Leftrightarrow \sigma \leq \sigma'.$$

(3) *We have the following.*

$$[w, (s_1 w) \vee \cdots \vee (s_{n-1} w)] = [w, (s_1 \vee \cdots \vee s_{n-1}) w] = \langle s_1, \dots, s_{n-1} \rangle w \stackrel{\text{poset}}{\simeq} (\mathfrak{S}_n, \leq).$$

Proof. We show (1). Suppose that there exists $\sigma \in \langle s_1, \dots, s_{n-1} \rangle$ and $j \in \{1, \dots, n-1\}$ such that $\sigma w \geq w$ and $s_j \sigma w \not\geq w$. Let $s_{i_\ell} \cdots s_{i_{n-i+2}} s_n \cdots s_i$ be a reduced expression of σw . We set $i_k := k + i - 1$ for $k \leq n - i + 1$ and $i_{\ell+1} = j$. Then by Theorem 2.14, there exists $(j < k)$ such that

$$(i) \quad s_{i_{\ell+1}} \cdots s_{i_{j+1}}(i_j) > s_{i_{\ell+1}} \cdots s_{i_{j+1}}(i_j + 1)$$

If $j \leq n - i + 1$, then $s_{i_{\ell+1}} \cdots s_{i_{j+1}}(i_j + 1) = n + 1$. This contradicts to (i). Thus $j > n - i + 1$. Then Theorem 2.14 says that $s_j \sigma w = s_{i_{\ell+1}} \cdots \widehat{s_{i_k}} \cdots \widehat{s_{i_j}} \cdots s_{i_1} s_n \cdots s_i$ and this expression have to be a reduced expression of $s_j \sigma w$. This contradicts to $s_j \sigma w \not\geq w$.

We prove (2). First we assume that $\sigma \leq \sigma'$ and show that $\sigma w \leq \sigma' w$. We may assume that $\sigma' = s_i \sigma$. By (1), there exists a reduced expression $s_{i_\ell} \cdots s_{i_1}$ of σ such that $s_{i_\ell} \cdots s_{i_1} s_n \cdots s_i$ is a reduced expression of σw . Then we want to show that $s_i s_{i_\ell} \cdots s_{i_1} s_n \cdots s_1$ is a reduced expression of $\sigma' w$. If not, then same argument used in the proof of (1) implies that there exists $j < k$ such that

$$s_i \sigma w = s_{i_{\ell+1}} s_{i_\ell} \cdots \widehat{s_{i_k}} \cdots \widehat{s_{i_j}} \cdots s_{i_1} w,$$

where we put $s_{i_{\ell+1}} := s_i$. This implies that $\ell(s_i \sigma) < \ell(\sigma)$. Hence we reach a contradiction.

Next we assume that $\sigma w \leq \sigma' w$. Then assertion follows from (1). In fact, there exists a reduced expression $s_{i_\ell} \cdots s_{i_1}$ of σ such that $s_{i_\ell} \cdots s_{i_1} s_n \cdots s_i$ is a reduced expression of σw . If we take a path

$$s_{i_\ell} \cdots s_{i_1} s_n \cdots s_1 \rightarrow s_{i_{\ell+1}} s_{i_\ell} \cdots s_{i_1} w \rightarrow \cdots \rightarrow s_{i_{\ell+\ell'}} \cdots s_{i_{\ell+1}} s_{i_\ell} \cdots s_{i_1} w = \sigma' w.$$

Then $s_{i_{\ell+\ell'}} \cdots s_{i_{\ell+1}} s_{i_\ell} \cdots s_{i_1}$ is a reduced expression of σ' .

We show the assertion (3). Let $J = \{1, \dots, n-1\}$. By Proposition 2.16, we have that

$$\bigvee_{j \in J} (s_j w) = (\bigvee_{j \in J} s_j) w = w_0(J) w.$$

Let $w' \in [w, w_0(J)w]$. Then there is a path

$$w \leftarrow s_{i_1} w \leftarrow \dots \leftarrow s_{i_{\ell'}} \dots s_{i_1} w = w' \leftarrow \dots \leftarrow s_{i_\ell} \dots s_{i_{\ell'+1}} s_{i_{\ell'}} \dots s_{i_1} w = w_0(J) w.$$

Let $\sigma = s_{i_{\ell'}} \dots s_{i_1}$. By definition, $s_{i_{\ell'}} \dots s_{i_1}$ is a reduced expression of σ and $s_{i_\ell} \dots s_{i_1}$ is a reduced expression of $w_0(J)$. Also we have that $\sigma \leq w_0(J)$. Hence Proposition 2.16(2) implies that $w' \in \langle s_j \mid j \in J \rangle w$. Then the assertion follows from (1) and (2). \square

Similarly, we have the following.

Lemma 4.19. *Let $w = s_1 s_2 \dots s_i$.*

(1) *For any $\sigma \in \langle s_2, \dots, s_n \rangle$, we have*

$$\sigma w \geq w.$$

(2) *Let $\sigma, \sigma' \in \langle s_2, \dots, s_n \rangle$. Then*

$$\sigma w \leq \sigma' w \Leftrightarrow \sigma \leq \sigma'.$$

(3) *We have the following.*

$$[w, (s_2 w) \vee \dots \vee (s_n w)] = [w, (s_2 \vee \dots \vee s_n) w] = \langle s_2, \dots, s_n \rangle w \stackrel{\text{poset}}{\simeq} (\mathfrak{S}_n, \leq).$$

Lemma 4.20. *Let $w_i^+ = s_n \dots s_i$. We put $M_i^+ \in \text{ind add } T_{w_i^+}$ such that $M_i^+ \notin \text{add } T_{s_n w_i^+}$. Then*

$$\text{s}\tau\text{-tilt}_{M_i^+} \Lambda = \rho([w_i^+, (s_1 w_i^+) \vee \dots \vee (s_{n-1} w_i^+)]) = \rho(\langle s_1, \dots, s_{n-1} \rangle w_i^+).$$

Proof. Since $s_j w_i^+ \geq w_i^+$ for any $j \in \{1, \dots, n-1\}$, we see that $T_{w_i^+}$ have to be the minimum element of $\text{s}\tau\text{-tilt}_{M_i^+} \Lambda$. (Note that $M_i^+ \in \text{add } T_{s_1 w_i^+} \cap \dots \cap \text{add } T_{s_{n-1} w_i^+}$.) Let T be a maximum element of $\text{s}\tau\text{-tilt}_{M_i^+} \Lambda$. Then we obtain that

$$T_{s_1 w_i^+} \vee \dots \vee T_{s_{n-1} w_i^+} \leq T.$$

In particular, Lemma 4.18 implies that

$$\text{s}\tau\text{-tilt}_{M_i^+} \Lambda \supset [T_{w_i^+}, T_{s_1 w_i^+} \vee \dots \vee T_{s_{n-1} w_i^+}] = \rho(\langle s_1, \dots, s_{n-1} \rangle w_i^+) \simeq (\mathfrak{S}_n, \leq).$$

Since (\mathfrak{S}_n, \leq) is a $(n-1)$ -regular poset, we conclude that

$$\text{s}\tau\text{-tilt}_{M_i^+} \Lambda = [T_{w_i^+}, T_{s_1 w_i^+} \vee \dots \vee T_{s_{n-1} w_i^+}] = \rho(\langle s_1, \dots, s_{n-1} \rangle w_i^+).$$

\square

Similarly we obtain the following.

Lemma 4.21. *Let $w_i^- = s_1 \dots s_i$. We put $M_i^- \in \text{ind add } T_{w_i^-}$ such that $M_i^- \notin \text{add } T_{s_1 w_i^-}$. Then*

$$\text{s}\tau\text{-tilt}_{M_i^-} \Lambda = \rho([w_i^-, (s_2 w_i^-) \vee \dots \vee (s_n w_i^-)]) = \rho(\langle s_2, \dots, s_n \rangle w_i^-).$$

Lemma 4.22. $\text{s}\tau\text{-tilt}_{P_{\sigma(i)}} \Lambda = \rho([\bigwedge_{k \neq i} (s_k w_0), w_0]) = \rho(\langle s_k \mid k \neq i \rangle w_0).$

Proof. Note that for any $k \neq i$, we have $P_{\sigma(i)} \in \mathbf{add} T_{s_k w_0}$. Let N_i be the minimum element of $\mathbf{s\tau\text{-tilt}}_{P_{\sigma(i)}} \Lambda$. Then $N_i \leq \bigwedge_{i \neq k} T_{s_k w_0}$ and

$$\mathbf{s\tau\text{-tilt}}_{P_{\sigma(i)}} \Lambda \supset [\bigwedge_{i \neq k} T_{s_k w_0}, \Lambda].$$

Now we let $\bigwedge_{i \neq k} (s_k w_0) = w w_0$ and $w' = \bigvee_{i \neq k} s_k$. Then $w' w_0 \leq s_k w_0$ for any $k \neq i$. Thus we conclude that

$$w' w_0 \leq w w_0.$$

Therefore we obtain

$$w' \geq w.$$

On the other hands, we have $w w_0 \leq s_k w_0 (\Leftrightarrow w \geq s_k)$ for any $k \neq i$. In particular, $w \geq w'$. Hence we obtain $w = w'$ and

$$[\bigwedge_{k \neq i} (s_k w_0), w_0] = [(\bigvee_{k \neq i} s_k) w_0, w_0] = \langle s_k \mid k \neq i \rangle w_0 \simeq (\mathfrak{S}_i \times \mathfrak{S}_{n-i+1}, \leq^{\text{op}}).$$

Since $(\mathfrak{S}_i \times \mathfrak{S}_{n-i+1}, \leq^{\text{op}})$ is a $(n-1)$ -regular poset, we obtain the assertion. \square

Lemma 4.23. *We have the following.*

- (1) $M_1^+ = P_{\sigma(n)}$ and $M_n^- = P_{\sigma(1)}$.
- (2) If $i \neq 1$, then $M_i^+ = e_i \Lambda / e_i \Lambda (1 - e_i - \dots - e_n) \Lambda$. Furthermore, we have that

$$\mathbf{s\tau\text{-tilt}}_{M_i^+} \Lambda \cap \mathbf{s\tau\text{-tilt}}_{P_{\sigma(n-i+1)}} \Lambda \neq \emptyset.$$

- (3) If $i \neq n$, then $M_i^- = e_i \Lambda / e_i \Lambda (1 - e_i - \dots - e_1) \Lambda$. Moreover, we have that

$$\mathbf{s\tau\text{-tilt}}_{M_i^-} \Lambda \cap \mathbf{s\tau\text{-tilt}}_{P_{\sigma(n-i+1)}} \Lambda \neq \emptyset.$$

Proof. We show (1). Note that $w_0 = s_1(s_2 s_1) \dots (s_n \dots s_1) \in \langle s_1, \dots, s_{n-1} \rangle w_1^+$ and $w_1^+ = s_n \dots s_1 \in \langle s_1, \dots, s_{n-1} \rangle w_0$. We also note that $w_0 = (s_n)(s_{n-1} s_n) \dots (s_1 \dots s_n) \in \langle s_2, \dots, s_n \rangle w_n^-$ and $w_n^- = s_1 \dots s_n \in \langle s_2, \dots, s_n \rangle w_0$. Hence we have $\langle s_1, \dots, s_{n-1} \rangle w_1^+ = \langle s_1, \dots, s_{n-1} \rangle w_0$ and $\langle s_2, \dots, s_n \rangle w_n^- = \langle s_2, \dots, s_n \rangle w_0$. Then the assertion follows from Lemma 4.20, Lemma 4.21 and Lemma 4.22. In fact, we see that

$$\mathbf{s\tau\text{-tilt}}_{M_1^+} \Lambda = \mathbf{s\tau\text{-tilt}}_{P_{\sigma(n)}} \Lambda, \quad \mathbf{s\tau\text{-tilt}}_{M_n^-} \Lambda = \mathbf{s\tau\text{-tilt}}_{P_{\sigma(1)}} \Lambda.$$

Next we prove (2). We claim that

$$\langle s_k \mid k \neq n-i+1 \rangle w_0 = \{w \in \mathfrak{S}_{n+1} \mid w(a) \leq n-i+1 \text{ for any } a \geq i+1\}.$$

Since $w_0(a) = n-a+2$, we obtain

$$\langle s_k \mid k \neq n-i+1 \rangle w_0 \subset \{w \in \mathfrak{S}_{n+1} \mid w(a) \leq n-i+1 \text{ for any } a \geq i+1\}.$$

Then

$$\langle s_k \mid k \neq n-i+1 \rangle w_0 = \{w \in \mathfrak{S}_{n+1} \mid w(a) \leq n-i+1 \text{ for any } a \geq i+1\}.$$

follows from the fact that

$$\langle s_k \mid k \neq n-i+1 \rangle w_0 \xleftrightarrow{1:1} \mathfrak{S}_i \times \mathfrak{S}_{n-i+1} \xleftrightarrow{1:1} \{w \in \mathfrak{S}_{n+1} \mid w(a) \leq n-i+1 \text{ for any } a \geq i+1\}.$$

Let $w = (s_{n-1} \dots s_1)(s_{n-1} \dots s_2) \dots (s_{n-1} \dots s_{i-1}) w_i^+$. One can easily check that

$$w(a) \leq n-i+1$$

for any $a \geq i + 1$. Hence $w \in \langle s_k \mid k \neq n - i + 1 \rangle w_0 \cap \langle s_1, \dots, s_{n-1} \rangle w_i^+$. Then the assertion follows from Lemma 4.17 (1), Lemma 4.20 and Lemma 4.22.

Similar argument implies the assertion (3). \square

Lemma 4.24. *We have the following.*

- (1) $\sigma(i) = n - i + 1$.
- (2) $\text{s}\tau\text{-tilt}_{P_n^-} \Lambda = \rho(\langle s_1, \dots, s_{n-1} \rangle)$.
- (3) $\text{s}\tau\text{-tilt}_{P_1^-} \Lambda = \rho(\langle s_2, \dots, s_n \rangle)$.
- (4) $\text{Supp}(P_1) = \text{Supp}(P_n) = Q_0$.

Proof. We prove (1). We first consider the case that n is odd. Let $i = \frac{n+1}{2}$. By Lemma 4.15 (4), either (i) $\sigma(a) = n + 1 - a$ for any $a \in \{1, \dots, n\}$ or (ii) $\sigma(a) = a$ for any $a \in \{1, \dots, n\}$. occurs. In particular, we have $\sigma(i) = i$. Now it is sufficient to show that $\sigma(i - 1) = i + 1$. If not, then we have $\sigma(i - 1) = i - 1$. Let a minimal projective presentation

$$P_{M_{i+1}^+} := [P_i^r \rightarrow P_{i+1}(\rightarrow e_{i+1}\Lambda/e_{i+1}\Lambda e_i\Lambda = M_{i+1}^+)]$$

of M_{i+1}^+ . By Lemma 4.23, we conclude that $M_{i+1}^+ \oplus P_{\sigma(n-i)} = M_{i+1}^+ \oplus P_{i-1}$ is τ -rigid. Therefore $\text{Hom}_{\mathbf{K}^b(\text{proj } \Lambda)}(P_{M_{i+1}^+}, P_{i-1}[1]) = 0$. This implies that $\alpha_{i-1} \in e_{i-1}\Lambda e_i$ factors through $i + 1$. (Note that $r > 0$.) Accordingly, we reach a contradiction.

Assume that n is even and let $i = \frac{n}{2}$. It is sufficient to show that $\sigma(i) = n - i + 1 = i + 1$. If not, then $\sigma(i) = i$. Consider a minimal projective presentation

$$P_{M_{i+1}^+} := [P_i^r \rightarrow P_{i+1}(\rightarrow e_{i+1}\Lambda/e_{i+1}\Lambda e_i\Lambda = M_{i+1}^+)]$$

of M_{i+1}^+ . Then as in the case that n is odd, we see that $\text{Hom}_{\mathbf{K}^b}(P_{M_{i+1}^+}, P_i[1]) = 0$. This is a contradiction.

We prove (2). Note that $w := s_1(s_2s_1) \cdots (s_{n-1} \cdots s_1) = s_1 \vee \cdots \vee s_{n-1}$ is a maximum element of $\langle s_1, \dots, s_{n-1} \rangle$. Let M be a maximum element of $\text{s}\tau\text{-tilt}_{P_n^-} \Lambda$. Then $M \geq T_{s_1} \vee \cdots \vee T_{s_{n-1}} = T_w$. Since $[0, T_w] = \rho([1, w]) = \rho(\langle s_1, \dots, s_{n-1} \rangle) \simeq (\mathfrak{S}_n, \leq)$, one obtains that

$$\text{s}\tau\text{-tilt}_{P_n^-} \Lambda = [0, T_w] = \rho(\langle s_1, \dots, s_{n-1} \rangle).$$

Similarly, one sees the assertion (3).

We show (4). $\text{Supp}(P_1) \neq Q_0$ implies that (P_1, P_n) is a τ -rigid pair. In particular, we have that $\text{s}\tau\text{-tilt}_{P_1} \Lambda \cap \text{s}\tau\text{-tilt}_{P_n^-} \Lambda \neq \emptyset$. By Lemma 4.22 and (1), one obtains that

$$\text{s}\tau\text{-tilt}_{P_1} \Lambda = \rho(\langle s_1, \dots, s_{n-1} \rangle w_0).$$

On the other hand, the assertion (2) of this Lemma implies that

$$\text{s}\tau\text{-tilt}_{P_n^-} \Lambda = \rho(\langle s_1, \dots, s_{n-1} \rangle).$$

Note that for any element $w \in \langle s_1, \dots, s_{n-1} \rangle w_0$, we have $w(n + 1) \neq n + 1$. Also note that for any element $w \in \langle s_1, \dots, s_{n-1} \rangle$, we have $w(n + 1) = n + 1$. This shows that

$$\langle s_1, \dots, s_{n-1} \rangle w_0 \cap \langle s_1, \dots, s_{n-1} \rangle = \emptyset.$$

We conclude that

$$\text{Supp}(P_1) = Q_0.$$

Similar argument implies that

$$\text{Supp}(P_n) = Q_0.$$

□

Lemma 4.25. *We have the following.*

- (1) $P_1 \oplus X_1$, $P_n \oplus X_n$ are τ -rigid and $P_i \oplus M_i^\pm$ is τ -rigid for any $i \neq 1, n$.
 (2) For any $i \neq 1, n$, we have a minimum projective presentation

$$P_{i-1} \xrightarrow{\alpha_{i-1}^*} P_i \rightarrow e_i \Lambda / e_i \Lambda e_{i-1} \Lambda = M_i^+$$

of $e_i \Lambda / e_i \Lambda e_{i-1} \Lambda$ and a minimum projective presentation

$$P_{i+1} \xrightarrow{\alpha_i} P_i \rightarrow e_i \Lambda / e_i \Lambda e_{i+1} \Lambda = M_i^-$$

of $e_i \Lambda / e_i \Lambda e_{i+1} \Lambda$. Furthermore, we obtain that

$$\alpha_{i-1}^* \Lambda = e_i \Lambda e_{i-1} \Lambda, \quad e_i \Lambda \alpha_{i-1}^* = e_i \Lambda e_{i-1}, \quad \alpha_i \Lambda = e_i \Lambda e_{i+1} \Lambda \text{ and } e_i \Lambda \alpha_i = e_i \Lambda e_{i+1}.$$

- (3) We have a minimum projective presentation

$$P_2 \xrightarrow{\alpha_1} P_1 \rightarrow e_1 \Lambda / e_1 \Lambda e_2 \Lambda = X_1$$

of X_1 . Moreover, we obtain that

$$\alpha_1 \Lambda = e_1 \Lambda e_2 \Lambda \text{ and } e_1 \Lambda \alpha_1 = e_1 \Lambda e_2.$$

- (4) We have a minimum projective presentation

$$P_{n-1} \xrightarrow{\alpha_{n-1}^*} P_n \rightarrow e_n \Lambda / e_n \Lambda e_{n-1} \Lambda = X_n$$

of X_n . Furthermore, we obtain that

$$\alpha_{n-1}^* \Lambda = e_n \Lambda e_{n-1} \Lambda \text{ and } e_n \Lambda \alpha_{n-1}^* = e_n \Lambda e_{n-1}.$$

Proof. We prove (1). By Lemma 4.24 (2), we see that $n \notin \text{Supp}(T_{s_n w_1^+})$ and $n \in \text{Supp}(T_{w_1^+})$. This implies that $T_{w_1^+} = T_{s_n w_1^+} \oplus M_1^+$ (see definition of M_1^+). By Lemma 4.23 and Lemma 4.24, we obtain that $M_1^+ = P_1$. In particular, $\text{add}(T_{s_n w_1^+} \oplus P_1) \ni X_1 \oplus P_1$ is τ -rigid. Similarly, we can check that $P_n \oplus X_n$ is τ -rigid. Also $P_i \oplus M_i^\pm$ are τ -rigid by Lemma 4.23 (2), (3) and Lemma 4.24 (1).

We show (2). Let

$$\bigoplus_{t=1}^r P_{i-1}^{(t)} = P_{i-1}^r \xrightarrow{f} P_i \rightarrow e_i \Lambda / e_i \Lambda e_{i-1} \Lambda = M_i^+$$

be a minimal projective presentation of $M_i^+ = e_i \Lambda / e_i \Lambda e_{i-1} \Lambda$. (1) implies that

$$\text{Hom}_{\mathbf{K}^b(\text{proj } \Lambda)}(P_{M_i^+}, P_i[1]) = 0.$$

Now we put $f = (f^{(t)} : P_{i-1}^{(t)} \rightarrow P_i)$ and consider $\phi \in \text{Hom}_{\mathbf{K}^b(\text{proj } \Lambda)}(P_{M_i^+}, P_i[1])$ given by $\varphi^{(t)} : P_{i-1}^{(t)} \rightarrow P_i$, where $\varphi^{(t)} = \begin{cases} \alpha_{i-1}^* & t = 1 \\ 0 & t \neq 1 \end{cases}$. Then there exists $h \in \text{End}_\Lambda(P_i)$ such that

$$(*) \quad h \circ f^{(t)} = \varphi^{(t)}$$

for any t . This shows that h has to be an isomorphism and $r = 1$. Let $x = f(e_{i-1})$ and $y = h(e_i)$. Then $x\Lambda = e_i\Lambda e_{i-1}\Lambda$ and $yx = \alpha_{i-1}^*$. Since $x\Lambda = e_i\Lambda e_{i-1}\Lambda$, there exists $y' \in e_{i-1}\Lambda e_{i-1} \setminus \text{Rad}(e_{i-1}\Lambda e_{i-1})$ such that $xy' = \alpha_{i-1}^*$. Hence we obtain

$$\alpha_{i-1}^*\Lambda = xy'\Lambda = x\Lambda = e_i\Lambda e_{i-1}\Lambda.$$

$\text{Hom}_{\mathbb{K}^b}(P_{M_i^+}, P_i[1]) = 0$ implies that for any morphism g from P_{i-1} to P_i , there exists $h' \in \text{End}_{\Lambda}(P_i)$ such that $g = h' \circ f$. This says that $e_i\Lambda e_{i-1} = e_i\Lambda x$. Therefore, we see that

$$e_i\Lambda\alpha_{i-1}^* = e_i\Lambda yx = e_i\Lambda x = e_i\Lambda e_{i-1}.$$

By applying same argument to the minimum projective presentation

$$\bigoplus_{t=1}^r P_{i+1}^{(t)} = P_{i+1}^r \xrightarrow{f} P_i \rightarrow e_i\Lambda/e_i\Lambda e_{i+1}\Lambda = M_i^-$$

of M_i^- , we have that $r = 1$ and

$$\alpha_i\Lambda = e_i\Lambda e_{i+1}\Lambda, \quad e_i\Lambda\alpha_i = e_i\Lambda e_{i+1}.$$

We now get the assertion (2).

Similarly one obtains (3) and (4). □

By Lemma 4.24 and Lemma 4.25, we have the following.

Proposition 4.26. *$\mathfrak{s}\tau$ -tilt $\Lambda \simeq (\mathfrak{S}_{n+1}, \leq)$ only if Λ satisfies the Condition 3.1.*

Proof. Condition 3.1 (a) follows from Lemma 4.14 (4) and Condition 3.1 (b) follows from Lemma 4.25. Hence it is sufficient to show that

$$\alpha_1 \cdots \alpha_{n-1} \neq 0 \neq \alpha_{n-1}^* \cdots \alpha_1^*.$$

If $\alpha_1 \cdots \alpha_{n-1} = 0$, then Lemma 4.25 implies that $n \notin \text{Supp}(P_1)$. This contradicts to Lemma 4.24 (4). Therefore, we obtain

$$\alpha_1 \cdots \alpha_{n-1} \neq 0.$$

Likewise, we also obtain

$$\alpha_{n-1}^* \cdots \alpha_1^* \neq 0.$$

□

5. SOME REMARKS ON G-VECTORS

In this section, we see that for two algebras satisfying the Condition 3.1, an poset isomorphism $\mathfrak{s}\tau$ -tilt Λ from $\mathfrak{s}\tau$ -tilt Γ preserves g-vectors.

Proposition 5.1. *Let ρ, ρ' be poset isomorphisms from $(\mathfrak{S}_{n+1}, \leq)$ to $\mathfrak{s}\tau$ -tilt Λ . If $\rho(s_i) = \rho'(s_i)$ holds for any i , then we have $\rho = \rho'$.*

Proof. We show the following claim.

Claim 6. *Let $w \in \mathfrak{S}_{n+1}$ and $s_{i_\ell} \cdots s_{i_1}$ a reduced expression of w . Assume that $\ell(s_j w) = \ell + 1$ and put $w' = s_j s_{i_{\ell-1}} \cdots s_{i_1}$. Then we have the following.*

- (a) *If $\ell(w') = \ell$, then $s_j w$ is a unique element of $\text{dp}(w) \cap [w, w \vee w']$.*
- (b) *If $\ell(w') = \ell - 2$, then $s_j w = s_{i_\ell} w' \vee s_j w'$.*

Proof. We show the assertion (a). In the case that $|i_\ell - j| > 1$, it is obvious that $s_j w = w \vee w'$. Thus we may assume that $|i_\ell - j| = 1$. Note that $\ell(s_{i_\ell} s_j w) = \ell(s_j s_{i_\ell} w') = \ell + 2$. If not, then $\ell(s_{i_\ell} s_j w) = \ell(s_j s_{i_\ell} w') = \ell$ and $s_{i_\ell} s_j w = s_j s_{i_\ell} w' = s_j w \wedge s_{i_\ell} w'$. Thus we have that $s_{i_{\ell-1}} \cdots s_{i_1} \leq s_j w, s_{i_\ell} w'$ and

$$s_{i_{\ell-1}} \cdots s_{i_1} < s_{i_\ell} s_j w.$$

By considering lengths, we see that there exists k such that $s_k s_{i_{\ell-1}} \cdots s_{i_1} = s_{i_\ell} s_j w$. Hence $s_k = s_{i_\ell} s_j s_{i_\ell}$, this is a contradiction.

Therefore, there are two paths

$$s_{i_\ell} s_j w \rightarrow s_j w \rightarrow w \rightarrow s_{i_{\ell-1}} \cdots s_{i_1} \text{ and } s_{i_\ell} s_j w = s_j s_{i_\ell} w' \rightarrow s_{i_\ell} w' \rightarrow w' \rightarrow s_{i_{\ell-1}} \cdots s_{i_1}.$$

This gives the assertion (a).

Next we show the assertion (b). Since $\ell(w') = \ell - 2$, we have that $|i_\ell - j| = 1$ and $\ell(s_j w) = \ell(s_{i_\ell} s_j s_{i_\ell} w') = \ell + 1$. Then we have two paths

$$s_j w \rightarrow w \rightarrow s_{i_{\ell-1}} \cdots s_{i_1} = s_j w' \rightarrow w' \text{ and } s_j w = s_{i_\ell} s_j s_{i_\ell} w' \rightarrow s_j s_{i_\ell} w' \rightarrow s_{i_\ell} w' \rightarrow w'.$$

This implies the assertion (b). ■

Claim 6 says that an poset automorphism φ is uniquely determined by $\varphi(s_1), \dots, \varphi(s_n)$. In, particular, if $\varphi(s_i) = s_i$ holds for any i , then $\varphi = \text{id}$. This gives the assertion. □

Corollary 5.2. *Let $\Lambda = kQ/I$, $\Gamma = kQ'/I'$ be algebras satisfying the Condition 3.1. Assume that Q° and $(Q')^\circ$ are the double quiver of $1 \rightarrow 2 \rightarrow \cdots \rightarrow n$. Then there is a unique poset isomorphism $\rho : \tau\text{-tilt } \Lambda \xrightarrow{\sim} \tau\text{-tilt } \Gamma$ satisfying $\rho(e_i \Lambda / e_i \Lambda (1 - e_i) \Lambda) = e_i \Gamma / e_i \Gamma (1 - e_i) \Gamma$. Moreover, ρ preserves g -vectors i.e. we have that*

$$g^T = g^{\rho(T)},$$

for any $T \in \tau\text{-tilt } \Lambda$.

Proof. By Proposition 4.11, the map $X_i(\Lambda) \rightarrow X_i(\Gamma)$ induces a desired poset isomorphism. Uniqueness follows from Proposition 5.1. □

REFERENCES

- [AIR] T. ADACHI, O. IYAMA, I. REITEN, τ -tilting theory. *Compos. Math.* **150**, no. 3, 415–452 (2014).
- [AI] T. AIHARA, O. IYAMA, Silting mutation in triangulated categories. *J. Lond. Math. Soc. (2)* **85** (2012), no. 3, 633–668.
- [AK] T. AIHARA, R. KASE, Algebras sharing the same support τ -tilting poset with tree quiver algebras. in preparation.
- [ASS] I. ASSEM, D. SIMSON, A. SKOWROŃSKI, Elements of the representation theory of associative algebras. Vol. 1. London Mathematical Society Student Texts **65**, Cambridge University Press (2006).
- [ARS] M. AUSLANDER, I. REITEN, S. SMALØ, Representation theory of artin algebras. Cambridge studies in advanced mathematics **36**, Cambridge University Press (1995).
- [BjB] A. BJÖRNER, F. BRENTI, Combinatorics of Coxeter groups, Graduate Texts in Mathematics, vol. **231**, Springer, New York, 2005.
- [BrB] S. BRENNER, M.C.R. BUTLER, Generalizations of the Bernstein-Gelfand-Ponomarev reflection functors. Representation theory, II (Proc. Second Internat. Conf., Carleton Univ., Ottawa, Ont., 1979), pp.103-169, Lecture Notes in Math., **832**, Springer, Berlin-New York (1980).
- [BIRS] A. B. BUAN, O. IYAMA, I. REITEN, J. SCOTT, Cluster structures for 2-Calabi-Yau categories and unipotent groups. *Compos. Math.* **145**, no. 4, 1035–1079 (2009).

- [EJR] F. EISELE, G. JANSSENS, T. RAEDSCHELDERS, A reduction theorem for τ -rigid modules. arXiv:1603.04293.
- [H] D. HAPPEL, Triangulated categories in the representation theory of finite-dimensional algebras. London Mathematical Society Lecture Note Series, **119**. Cambridge University Press, Cambridge (1988).
- [HU] D. HAPPEL, L. UNGER, On the quiver of tilting modules. *J. Algebra* **284**, no. 2, 857–868 (2005).
- [IZ] O. IYAMA, X. ZHANG Classifying τ -tilting modules over the Auslander algebras of $K[X]/(X^n)$. arXiv:1602.05037
- [J] G. JASSO, Reduction of τ -tilting modules and torsion pairs. *Int. Math. Res. Not. IMRN 2015*, no. 16, 7190–7273.
- [M] Y. MIZUNO Classifying τ -tilting modules over preprojective algebras of Dynkin type *Math. Z.* **277**, no. 3-4, 665–690 (2014).
- [RS] C. RIEDTMANN, A. SCHOFIELD, On a simplicial complex associated with tilting modules. *Comment. Math. Helv.* **66**, no. 1, 70–78 (1991).

DEPARTMENT OF MATHEMATICS, NARA WOMEN'S UNIVERSITY, KITAUOYA-NISHIMACHI, NARA CITY, NARA 630-8506, JAPAN

E-mail address: r-kase@cc.nara-wu.ac.jp